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ON STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Let E denote the unit disc and $H(E)$ the class of all holomorphic functions f on E with $f(0) = 0 = f'(0) - 1$. The class S_s^* of starlike functions with respect to symmetric points consists of functions $f \in H(E)$ satisfying the condition:

$$\text{Re}\,[zf'(z)/(f(z) - f(-z))] > 0, \mid z \mid < 1.$$

We define two sub-classes,

$$R(\alpha) = \{f : \mid (F(z) - 1)/(F(z) + 1) \mid < \alpha, \ \ 0 < \alpha \leqslant 1\}$$

and

$$T(\alpha) = \{f : \mid (F(z) - \alpha) \mid < \alpha, \ \ \alpha > \tfrac{1}{2}\}$$

where

$$F(z) = 2zf'(z)/(f(z) - f(-z)).$$

The radius of convexity is determined for each class and other properties like coefficients estimates, behaviour of certain integral operators on the members of these classes, distortion theorems, representation theorems are studied.

1. INTRODUCTION

A function $f(z)$ regular in $E(\mid z \mid < 1)$ and with normalization

$$f(0) = 0 = f'(0) - 1$$

is said to be starlike with respect to symmetric points in E if

$$\text{Re}\left(\frac{zf'(z)}{(f(z) - f(-z))}\right) > 0, \mid z \mid < 1. \qquad ...(1.1)$$

Let S_s^* denote the class of such functions. This class was introduced by Sakaguchi (1959).

We define two sub-classes

$$R(\alpha) = \{f : \mid (F(z) - 1)/(F(z) + 1) \mid < \alpha, 0 < \alpha \leqslant 1\} \qquad ...(1.2)$$

$$T(\alpha) = \{f : \mid F(z) - \alpha \mid \langle \ \alpha, \alpha \ \rangle \ \tfrac{1}{2}\} \qquad ...(1.3)$$

where $\quad F(z) = 2zf'(z)/(f(z) - f(-z)).$

We observe that when $\alpha = 1$ condition (1.2) reduces to (1.1) and the class $R(\alpha)$ coincides with S_s^*. The same is true of the class $T(\alpha)$ when $\alpha = \infty$.

In this paper we obtain coefficient estimates, distortion theorems and radius of convexity for each class and also study the behaviour of certain integral operators on the members of these classes.

Some results of Sakaguchi (1959) and Das and Singh (1977a) are deduced as particular cases.

2. Coefficient Estimates

Theorem 2.1 — Let $f(z) = z + a_2 z^2 + \ldots + a_k z^k + \ldots \in R(\alpha)$.

Then for $n \geqslant 1$,

$$\left.\begin{aligned} |a_{2n}| &\leqslant \frac{1}{n!}\alpha(\alpha+1)\ldots(\alpha+n-1) \\ |a_{2n+1}| &\leqslant \frac{1}{n!}\alpha(\alpha+1)\ldots(\alpha+n-1). \end{aligned}\right\} \qquad \ldots(2.1.1)$$

For proving the theorem we use the following lemma.

Lemma A (Padmanabhan 1970) — Let $H(z) = 1 + \sum_{k=1}^{\infty} H_k z^k$ be analytic in E and satisfy $|(1 - H(z))/(1 + H(z))| < \alpha$, $0 < \alpha \leqslant 1$, $z \in E$. Then we have

$$H(z) = \frac{1 - \alpha w(z)}{1 + \alpha w(z)}$$

where $w(z)$ is analytic in E and $w(0) = 0$, $|w(z)| < 1$ in E and conversely. Further $|H_k| \leqslant 2\alpha$ for $k \geqslant 1$.

Proof of the Theorem : The function $\dfrac{2zf'(z)}{f(z) - f(-z)}$ satisfies the hypotheses of the lemma. Hence we have

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 - \alpha w(z)}{1 + \alpha w(z)} \qquad \ldots(2.1.2)$$

Setting $\dfrac{1 - \alpha w(z)}{1 + \alpha w(z)} = 1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots$ we have

$$z + 2a_2 z^2 + \ldots + (n+1)\, z^{n+1} + \ldots$$
$$= (z + a_3 z^3 + \ldots + a_{2n-1} z^{2n-1} + \ldots)(1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots).$$

Equating corresponding coefficients,

$$a_2 = \frac{b_1}{2};\ a_3 = \frac{b_2}{2};\ 4a_4 = b_1a_3 + b_3,\ 4a_5 = b_2a_3 + b_4$$

$$2ka_{2k} = b_{2k-1} + a_3b_{2k-3} + a_5b_{2k-5} + \dots + a_{2k-1}b_1$$

$$2ka_{2k+1} = b_{2k} + b_{2k-2}a_3 + b_{2k-2}a_5 + \dots + b_2a_{2k-1}.$$

By Lemma A, $|b_n| \leqslant 2\alpha$ for $n \geqslant 1$. Hence we have

$$|a_2| \leqslant \alpha,\ |a_3| \leqslant \alpha;\ |a_4| \leqslant \frac{1}{\alpha}(\alpha + 1),\ |a_5| \leqslant \tfrac{1}{2}\alpha(\alpha + 1).$$

We obtain by induction on k

$$2n\,|a_{2n}| \leqslant 2\alpha + 2\alpha\,.\,\alpha + 2\alpha\,.\,\frac{1}{2!} \times \alpha(\alpha + 1) + \frac{2\alpha}{3!}\alpha(\alpha + 1)$$

$$\times (\alpha + 2) \dots + \frac{2\alpha}{(n-1)!}\alpha(\alpha + 1) \dots (\alpha + n - 2)$$

$$\text{R.H.S.} = 2\alpha(\alpha + 1)\,.\,\frac{(\alpha + 2)}{2}\,.\,\frac{(\alpha + 3)}{3} \dots \frac{(\alpha + n - 2)}{n - 2}\left(1 + \frac{\alpha}{n - 1}\right).$$

Hence $$|a_{2n}| \leqslant \frac{1}{n!}\alpha(\alpha + 1) \dots (\alpha + n - 1)$$

Similarly $|a_{2n+1}| \leqslant \frac{1}{n!}\alpha(\alpha + 1) \dots (\alpha + n - 1).$

Theorem 2.2 — Let $f(z) = z + a_2z^2 + \dots + a_nz^n + \dots \in T(\alpha)$. Then for $n \geqslant 1$,

$$\left.\begin{aligned} |a_{2n}| &\leqslant \frac{(1 - A)(3 - A)\dots(2n - 1 - A)}{n!\,2^n} \\ |a_{2n+1}| &\leqslant \frac{(1 - A)(3 - A)\dots(2n - 1 - A)}{n!\,2^n} \end{aligned}\right\} \quad \dots(2.2.1)$$

where $A = (1 - \alpha)/\alpha$.

This theorem can be proved in a similar manner by making use of the following lemma.

Lemma B (Padmanabhan 1972) — Suppose $G(z) = 1 + \sum_{k=1}^{\infty} G_kz^k$ is analytic in E and $G(0) = 1$. If $|G(z) - \alpha| < \alpha$ where α is any real number greater than $\frac{1}{2}$, then $G(z) = (1 + w(z)/(1 + Aw(z))$ where $A = (1 - \alpha)/\alpha$, w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E and conversely. Further $|G_n| \leqslant 1 - A$, $n \geqslant 1$.

Remark : Putting $\alpha = 1$ in (2.1.1) and $\alpha = \infty$ or $A = -1$ in (2.2.1) we obtain $|a_n| \leqslant 1$ for $n \geqslant 1$ where $f(z) = z + \sum_{k=2}^{\infty} a_kz^k$ is starlike with respect to symmetric points (Sakaguchi 1959).

3. Radius of Convexity

Theorem 3.1 — Let $f \in R(\alpha)$. Then for $|z| = r$, $0 \leqslant r < 1$ we have

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) \geqslant \begin{cases} \dfrac{1-r^2}{1+r^2} + \dfrac{1}{\alpha(1-r^2)}[\sqrt{(1-\alpha^2)(1-\alpha^2 r^4)} - (1+\alpha^2 r^2)], & R_0 \geqslant R_1 \\ \dfrac{(1-r^2)(1-\alpha^2 r^2) - 2\alpha r(1+r^2)}{(1+r^2)(1-\alpha^2 r^2)}, & R_0 \leqslant R_1. \end{cases} \quad \text{...(3.1.1)}$$

Proof : By Lemma A, $\dfrac{2zf'(z)}{f(z) - f(-z)} = \dfrac{1 - \alpha w(z)}{1 + \alpha w(z)}$, $|w(z)| \leqslant 1$, $z \in E$.
Logarithmic differentiation and simplification yield

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} - \frac{2\alpha z w'(z)}{(1 - \alpha w(z))(1 + \alpha w(z))} \quad \text{...(3.1.2)}$$

where $g(z) = \frac{1}{2}(f(z) - f(-z))$ is odd and starlike with respect to the origin. Also

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \frac{1-r^2}{1+r^2}. \quad \text{...(3.1.3)}$$

We know that

$$|zw'(z) - w(z)| \leqslant \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}. \quad \text{...(3.1.4)}$$

Using (3.1.3) and (3.1.4) in (3.1.2) we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1-r^2}{1+r^2} - 2\alpha\left[\operatorname{Re}\left(\frac{w(z)}{(1+\alpha w(z))(1-\alpha w(z))}\right) + \frac{|z|^2 - |w(z)|^2}{(1-|z|^2)\,|1+\alpha w(z)|\,.\,|1-\alpha w(z)|}\right].$$

Following Singh and Goel (1971), consider the transformation

$$P(z) = (1 - \alpha w(z))/(1 + \alpha w(z))$$

which maps $|w(z)| \leqslant r$ onto $|P(z) - a| \leqslant d$ where $a = (1 + \alpha^2 r^2)/(1 - \alpha^2 r^2)$, $d = 2\alpha r/(1 - \alpha^2 r^2)$. We obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1-r^2}{1+r^2} + \frac{1}{2}\left[\operatorname{Re}\left(P(z) - \frac{1}{P(z)}\right) - \frac{\alpha^2 r^2 |1 + P(z)|^2 - |1 - P(z)|^2}{\alpha(1 - r^2)\,|P(z)|}\right]. \quad \text{...(3.1.5)}$$

If $P(z) = a + u + iv$, $R = |P(z)|$, denoting the right-hand side of (3.1.5) by $S(u, v)$ we get

$$S(u, v) = \frac{1-r^2}{1+r^2} + \frac{1}{2}\left[a + u - \frac{a+u}{R^2} - \frac{(1-\alpha^2 r^2)}{\alpha(1-r^2)}\cdot\frac{(d^2 - u^2 - v^2)}{R}\right]. \quad \text{...(3.1.6)}$$

Differentiating with respect to v,

$$\frac{\partial S(u, v)}{\partial v} = \frac{v}{2R^4} T(u, v)$$

where $$T(u, v) = 2(a + u) + [2R^3 + (d^2 - u^2 - v^2) R] \frac{(1 - \alpha^2 r)}{\alpha(1 - r^2)}$$

$$> 0.$$

Thus we see that $S(u, v)$ attains its minimum on each chord $u =$ const when $v = 0$. Hence the minimum of $S(u, v)$ in the disc $| P(z) - a | \leqslant d$ is attained on the diameter $v = 0$. Putting $v = 0$ in $S(u, v)$ we have

$$S(u, 0) = L(R) = \frac{1 - r^2}{1 + r^2} + \frac{1}{2\alpha(1 - r^2)} \Big[(1 + \alpha)(1 - \alpha r^2) R + \frac{(1 - \alpha)(1 + \alpha r^2)}{R} - 2a(1 - \alpha^2 r^2)\Big]$$

where $a - d \leqslant R \leqslant a + d$.

So $L'(R) = 0$ when $R = R_0 = \{(1 - \alpha)(1 + \alpha r^2)\}^{1/2}/\{(1 + \alpha)(1 - \alpha r^2)\}^{1/2}$

and this minimum equals

$$L(R_0) = \frac{1 - r^2}{1 + r^2} + \frac{1}{\alpha(1 - r^2)} [\{(1 - \alpha^2)(1 - \alpha^2 r^4)\}^{1/2} - a(1 - \alpha^2 r^2)]. \qquad \ldots(3.1.7)$$

We see that $R_0 < a + d$. But R_0 may not always be greater than $a - d$. In this case the minimum is attained at $R_1 = a - d$ and

$$L(R_1) = \frac{(1 - r^2)(1 - \alpha^2 r^2) - 2\alpha r(1 + r^2)}{(1 + r^2)(1 - \alpha^2 r^2)}. \qquad \ldots(3.1.8)$$

Therefore from (3.1.7) and (3.1.8) we get

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \begin{cases} \dfrac{1 - r^2}{1 + r^2} + \dfrac{1}{\alpha(1 - r^2)} [\{(1 - \alpha^2)(1 - \alpha^2 r^4)\}^{1/2} - (1 + \alpha^2 r^2)], & R_0 \geqslant R_1 \\ \dfrac{(1 - r^2)(1 - \alpha^2 r^2) - 2\alpha r(1 + r^2)}{(1 + r^2)(1 - \alpha^2 r^2)}, & R_0 \leqslant R_1. \end{cases}$$

The first estimate is attained for $g(z) = \dfrac{z}{1 + z^2}$ and $w(z) = \dfrac{z(z - t)}{(1 - zt)}$ where $| t | < 1$ and t is given by $\dfrac{1 - (1 - \alpha) rt - \alpha r^2}{1 - (1 + \alpha) rt + \alpha r^2} = R_0$.

The second one is attained for the same choice of $g(z)$ and $w(z) = z$. The radii of convexity are determined from the equations

$$\frac{1-r^2}{1+r^2}+\frac{1}{\alpha(1-r^2)}\left(\{(1-\alpha^2)(1-\alpha^2r^4)\}^{1/2}-(1+\alpha^2r^2)\right)=0$$

which reduces to

$$(1-\alpha)+r^2(1-\alpha)-(1+6\alpha+\alpha^2)r^4+(1-\alpha)r^6-\alpha(1-\alpha)r^8=0 \quad ...(3.1.9)$$

and $$1-2\alpha r-r^2(1+\alpha^2)-2\alpha r^3+\alpha^2r^4=0. \quad ...(3.1.10)$$

The two minimum values given by the above inequalities will be equal for such values of $\alpha(0<\alpha\leqslant 1)$ for which $R_0=R_1$. Also the resultant of the polynomials on the left-hand side of (3.1.9) and (3.1.10) when treated as a function of α, can be shown to have a root between 0 and 1. Let α_0 be the smallest positive root. Then we have the following theorem.

Theorem 3.2 — Let $f(z)\in R(\alpha)$. Then f is convex in $|z|<r_0$ where r_0 is the smallest positive root of

$$(1-\alpha)+(1-\alpha)r^2-(1+6\alpha+\alpha^2)r^4+(1-\alpha)r^6-\alpha(1-\alpha)r^8=0$$
$$\text{for } 0<\alpha\leqslant\alpha_0 \quad ...(3.2.1)$$

and

$$1-2\alpha r-r^2(1+\alpha^2)-2\alpha r^3+\alpha^2r^4=0 \text{ for } \alpha_0\leqslant\alpha\leqslant 1. \quad ...(3.2.2)$$

Theorem 3.3 — Let $f(z)\in T(\alpha)$. Then f is convex in $|z|<r_0$ where r_0 is the smallest positive root of the equation

$$\alpha-2\alpha r+(1-2\alpha)r^2+2(1-\alpha)r^3-(1-\alpha)r^4=0. \quad ...(3.3.1)$$

Case i — Let $\frac{1}{2}<\alpha\leqslant 1$.

PROOF: By Lemma B,

$$\frac{2zf'(z)}{f(z)-f(-z)}=\frac{1+w(z)}{1+Aw(z)},\ A=\frac{1-\alpha}{\alpha},$$

$w(z)$ is analytic in E, $w(0)=0$, $|w(z)|<1$ in E. Putting $g(z)=\frac{1}{2}(f(z)-f(-z))$ and proceeding as in Theorem 3.1, we can show that

$$1+\frac{zf''(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{(1-A)zw'(z)}{(1+w(z))(1+Aw(z))} \quad ...(3.3.2)$$

and

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geqslant\frac{1-r^2}{1+r^2}+(1-A)\left[\mathrm{Re}\left(\frac{w(z)}{(1+w(z))(1+Aw(z))}\right)\right.$$
$$\left.-\frac{|z|^2-|w(z)|^2}{(1-|z|^2)|1+w(z)||1+Aw(z)|}\right]. \quad (3.3.3)$$

By considering the transformation $P(z) = \frac{1 + Aw(z)}{1 + w(z)}$ which maps $| w(z) | \leqslant r$ onto $| P(z) - a | \leqslant d$ where $a = \frac{1 - Ar^2}{1 - r^2}$, $d = \frac{(1 - A) r}{1 - r^2}$ we obtain

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1 - r^2}{1 + r^2} + \frac{1 + A}{1 - A} - \frac{1}{1 - A}\left[\mathrm{Re}\left((P(z) + \frac{A}{P(z)}\right) + \frac{r^2 \,| P(z) - A |^2 - | 1 - P(z) |^2}{(1 - r^2) \,| P(z) |}\right]. \quad \ldots(3.3.4)$$

If $P(z) = Re^{i\theta}$ and the right-hand side of (3.3.4) is denoted by $S(R, \theta)$ we get

$$S(R, \theta) = \frac{1 - r^2}{1 + r^2} + \frac{1 + A}{1 - A} - \frac{1}{1 - A}\left[\left(R + \frac{A}{R} + 2a\right)\cos\theta - R - \frac{a^2 - d^2}{R}\right]$$

$$\frac{\partial S(R, \theta)}{\partial \theta} = T(R) \sin\theta$$

where $T(R) = \frac{1}{1 - A}\left(R + \frac{1}{R} + 2a\right) > 0$ since $\frac{1}{2} < \alpha \leqslant 1$ implies $0 \leqslant A < 1$. Therefore $S(R, \theta)$ attains its minimum when $\theta = 0$ and

$$S(R, 0) = \frac{1 - r^2}{1 + r^2} + \frac{1 + A}{1 - A} - \frac{1}{1 - A}\left[2a - \frac{a^2 - d^2 - A}{R}\right]$$

$$a^2 - d^2 - A = \frac{(1 + Ar^2)(1 - A)}{1 - r^2} > 0, \; 0 \leqslant A < 1.$$

Thus $S(R, 0)$ is a decreasing function of R for $0 \leqslant A < 1$ and the minimum is attained at $R = a + d$ and equals

$$\frac{1 - r - 3r^2 - r^3 + A(r^2 + 3r^3 + r^4 - r^5)}{(1 - r^4)(1 - Ar)}. \quad \ldots(3.3.5)$$

Putting $A = (1 - \alpha)/\alpha$, from (3.3.4) and (3.3.5) we get

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{\alpha - 2\alpha r + (1 - 2\alpha) r^2 + 2(1 - \alpha) r^3 - (1 - \alpha) r^4}{(1 + r^2)(1 - r)(\alpha - (1 - \alpha) r)}$$

The bound is sharp for the function given by $g(z) = \frac{z}{1 + z^2}$ and $w(z) = -z$.

Case ii — Let $\alpha > 1$.

Putting $A = (1 - \alpha)/\alpha$, $w(z) = z\phi(z)$ in (3.3.2) where $\phi(z)$ is analytic and $| \phi(z) | < 1$ in E we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} - (2\alpha - 1)\left[\frac{z\phi(z) + z^2\phi'(z)}{(1 + z\phi(z))(\alpha + (1 - \alpha)\phi(z))}\right] \quad \ldots(3.3.6)$$

It is known that

$$| z\phi(z) + z^2\phi'(z) | \leqslant | z\phi(z) | + \frac{| z |^2 (1 - | \phi(z) |^2)}{1 - | z |^2}$$

and when $\alpha > 1$

$$| (1 + z\phi(z)) (\alpha + (1 - \alpha) \phi(z)) | \geqslant \alpha - | z\phi(z) | -(\alpha - 1) | z\phi(z) |^2.$$

Using these estimates in (3.3.6) we get

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \frac{1 - r^2}{1 + r^2} - \frac{(2\alpha - 1) [t(1 - r^2) + r^2 - t^2]}{(1 - r^2) [\alpha - t + (1 - \alpha) t^2]}$$

where $| z\phi(z) | = t$ and $| z | = r$. $0 \leqslant r < 1$ and $0 \leqslant t < r$. Therefore

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ if}$$

$$t^2(\alpha + (4\alpha - 3) r^2 + (1 - \alpha) r^4) - t(2\alpha - 2r^2 + 2(1 - \alpha) r^4) + (\alpha - (4\alpha - 1) r^2 + (1 - \alpha) r^4) > 0. \quad \ldots(3.3.7)$$

If $Q(t)$ denotes the left member of (3.3.7) we see $Q'(t) = 0$ when

$$t = t_1 = (\alpha - r^2 + (1 - \alpha) r^4)/(\alpha + (4\alpha - 3) r^2 + (1 - \alpha) r^4).$$

It is easy to check that t_1 is positive for $\alpha > 1$. Also $Q''(t) > 0$. Now $t_1 \gtreqless r$ according as

$$(\alpha - 1) r^5 - (\alpha - 1) r^4 - (4\alpha - 3) r^3 - r^2 - \alpha r + \alpha \gtreqless 0.$$

Denoting the left member of the above by $E(r)$ we see $E(r)$ has atleast one root in (0, 1). Denote by r_1 the smallest positive root. $E(r) > 0$ for $0 \leqslant r < r_1$. This meas for $0 \leqslant r < r_1$, $t_1 > r$ and $Q(t)$ attains its minimum at $t = r$ for $0 \leqslant t \leqslant r < r_1$. Now $Q(r) > 0$ implies $Q(t) > 0$ for $0 \leqslant t \leqslant r$. Thus the condition becomes

$$\alpha - 2\alpha r + (1 - 3\alpha) r^2 + 2r^3 + (3\alpha - 2) r^4 - 2(1 - \alpha) r^5 + (1 - \alpha) r^6 > 0$$

which reduces to

$$(1 - r) (1 + r) (\alpha - 2\alpha r + (1 - 2\alpha) r^2 + 2(1 - \alpha) r^3 - (1 - \alpha) r^4) > 0. \quad \ldots(3.3.8)$$

Denoting the left-hand side of (3.3.8) by $F(r)$ we see $F(r) = 0$ has atleast one root in (0, 1). Let r_0 be the smallest positive root. $F(r) > 0$ for $0 \leqslant r < r_0$ and

$$F(r_1) = (1 - r_1) r_1^2 (2 - 4\alpha) < 0 \text{ since } \alpha > 1/2.$$

Hence $r_0 < r_1$.

Remark : Putting $\alpha = 1$ in (3.2.2) and $\alpha = \infty$ in (3.3.8) we see that $f(z) \in S_s^*$ is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation

$$1 - 2r - 2r^2 - 2r^3 + r^4 = 0, \; r_0 = \tfrac{1}{2}((1 + \sqrt{5}) - \sqrt{2(1 + \sqrt{5})}).$$

The result is sharp for the function $\log \dfrac{1+z}{\sqrt{1+z^2}}$ (Das and Singh 1977a).

4. Distortion Theorems

Theorem 4.1 — Let $f(z) \in R(\alpha)$. Then for $|z| = r, 0 \leqslant r < 1$ we have

$$\left.\begin{aligned} \frac{1 - \alpha r}{(1 + r^2)(1 + \alpha r)} \leqslant |f'(z)| \leqslant \frac{1 + \alpha r}{(1 - r^2)(1 - \alpha r)} \\ \int_0^r \frac{(1 - \alpha t)\, dt}{(1 + t^2)(1 + \alpha t)} \leqslant |f(z)| \leqslant \int_0^r \frac{1 + \alpha t}{(1 - t^2)(1 - \alpha t)}\, dt. \end{aligned}\right\} \quad \ldots(4.1.1)$$

The above estimates are sharp.

Proof : By Lemma A,

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 - \alpha w(z)}{1 + \alpha w(z)}. \quad \ldots(4.1.2)$$

Putting $g(z) = \dfrac{f(z) - f(-z)}{2}$, since $g(z)$ is odd and starlike with respect to the origin, we know

$$\frac{r}{1 + r^2} \leqslant |g(z)| \leqslant \frac{r}{1 - r^2};\; |z| = r, 0 < r < 1.$$

Also

$$\frac{1 - \alpha |z|}{1 + \alpha |z|} \leqslant \left|\frac{1 - \alpha w(z)}{1 + \alpha w(z)}\right| \leqslant \frac{1 + \alpha |z|}{1 - \alpha |z|}$$

Using the above estimates in (4.1.2) we obtain

$$\frac{(1 - \alpha r)}{(1 + r^2)(1 + \alpha r)} \leqslant |f'(z)| \leqslant \frac{1 + \alpha r}{(1 - r^2)(1 - \alpha r)}$$

and

$$\int_0^r \frac{1 - \alpha t}{(1 + t^2)(1 + \alpha t)}\, dt \leqslant |f(z)| \leqslant \int_0^r \frac{1 + \alpha t}{(1 - t^2)(1 - \alpha t)}\, dt.$$

The extremal function corresponding to the left and right side of the above inequalities are attained for

$$f(z) = \int_0^z \frac{1 - \alpha t}{(1 + t^2)(1 + \alpha t)} dt \quad \text{and} \quad f(z) = \int_0^z \frac{1 + \alpha t}{(1 - t^2)(1 - \alpha t)} dt,$$

In a similar manner we can prove the following theorem.

Theorem 4.2 — Let $f(z) \in T(\alpha)$. Then for $|z| = r$, $0 < r < 1$

$$\left\{\begin{array}{l} \dfrac{\alpha(1-r)}{(\alpha + (1-\alpha) r)(1 + r^2)} \leqslant | f'(z) \leqslant \dfrac{\alpha(1+r)}{(\alpha - (1-\alpha) r)(1 - r^2)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tfrac{1}{2} < \alpha \leqslant 1 \\ \displaystyle\int_0^r \frac{\alpha(1-t)\, dt}{(\alpha + (1-\alpha) t)(1+t^2)} \leqslant | f(z) | \leqslant \int_0^r \frac{\alpha(1+t)}{(\alpha - (1-\alpha) t)(1-t^2)} dt, \end{array}\right. \qquad \ldots(4.2.1)$$

and

$$\left\{\begin{array}{l} \dfrac{\alpha(1-r)}{(\alpha + (\alpha-1) r)(1 + r^2)} \leqslant | f'(z) | \leqslant \dfrac{\alpha}{(\alpha - (\alpha-1) r)(1 - r)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha > 1 \\ \displaystyle\int_0^r \frac{\alpha(1-t)\, dt}{(\alpha + (\alpha-1) t)(1+t^2)} \leqslant | f(z) | \leqslant \int_0^r \frac{\alpha\, dt}{(\alpha - (\alpha-1) t(1-t)}. \end{array}\right. \qquad \ldots(4.2.2)$$

The estimates are sharp.

Remark : Putting $\alpha = 1$ in (4.1.1) and $\alpha = \infty$ in (4.2.2) we obtain the following distortion theorem for the class S_s^* ·

$$\left.\begin{array}{r} \dfrac{1-r}{(1+r)(1+r^2)} \leqslant | f'(z) | \leqslant \dfrac{1}{(1-r)^2} \\ \log \dfrac{1+r}{\sqrt{1+r^2}} \leqslant | f(z) | \leqslant \dfrac{r}{1-r}. \end{array}\right\} \text{(Das and Singh 1977a)}$$

5. Integral Operator on the Members of $R(\alpha)$ and $T(\alpha)$

We require the following lemmas.

Lemma 1 — Let $N(z)$ be analytic and $D(z)$ starlike in E and $N(0) = D(0) = 0$ $N'(0) - 1 = D'(0) - 1$. Then

$$| (N(z)/D(z)) - 1 | \,/\, | (N(z)/D(z)) + 1 | < \alpha, \; 0 < \alpha \leqslant 1 \text{ for } z \in E$$

whenever

$$\frac{| (N'(z)/D'(z)) - 1 |}{| (N'(z)/D'(z)) + 1 |} < \alpha, \; z \in E.$$

PROOF : Define $w(z)$ by $N(z)/D(z) = (1-\alpha w(z))/(1+\alpha w(z))$. Then $w(z)$ is analytic in E and $w(0) = 0$. If we prove $|w(z)| < 1$ in E, then by Lemma A it will follow that $\left|\frac{N(z)}{D(z)} - 1\right| \Big/ \left|\frac{N(z)}{D(z)} + 1\right| < \alpha$. Let $|w(z)| \nless 1, z \in E$. Then by Jack's (1971) lemma there exists z_0 such that $|z_0| < 1$, $|w(z_0)| = 1$, $z_0 w'(z_0) = kw(z_0)\ k \geqslant 1$. Now,

$$\left|\frac{\dfrac{N'}{D'} - 1}{\dfrac{N'}{D'} + 1}\right| = \alpha \left|\frac{w(z_0) + \dfrac{kw(z_0)}{1+\alpha w(z_0)}\,\dfrac{D(z_0)}{z_0 D'(z_0)}}{1 - \dfrac{k\alpha w(z_0)}{1+\alpha w(z_0)} \cdot \dfrac{D(z_0)}{z_0 D'(z_0)}}\right|$$

$$= \alpha \left|\frac{1 + P(z_0)}{1 - \alpha w(z_0)\,P(z_0)}\right| \text{ where } P(z_0) = \frac{k}{1+\alpha w(z_0)} \times \frac{D(z_0)}{z_0 D'(z_0)}$$

$$> \alpha$$

if $|1 + P(z_0)| > |1 - \alpha w(z_0)\,P(z_0)|$

that is, if $(1 - \alpha^2)\,|P(z_0)|^2 > -2\,\mathrm{Re}\,(1 + \alpha w(z_0))\,P(z_0)$.

This is true if $\mathrm{Re}\,(1 + \alpha w(z_0))\,P(z_0) > 0$ or $\mathrm{Re}\,k\,\dfrac{D(z_0)}{z_0 D'(z_0)} > 0$

which is true since $D \in S^*$ and $k > 0$. Thus we get a contradiction to the hypothesis and the lemma is proved.

Lemma 2 — If N and D are defined as in Lemma 1 and if $\left|\dfrac{N'}{D'} - \alpha\right| < \alpha,\ \alpha > \frac{1}{2}$, then $\left|\dfrac{N}{D} < \alpha\right| < \alpha$. This can be similarly proved by defining

$$\frac{N(z)}{D(z)} = \frac{1 + w(z)}{1 + Aw(z)},\ A = \frac{1-\alpha}{\alpha} \text{ and using Lemma B.}$$

Theorem 5.1 — Let $f(z) \in R(\alpha)$. Then $F(z)$ defined by $F(z) = \dfrac{2}{z}\displaystyle\int_0^z f(t)\,dt$ belongs to $R(\alpha)$ as well.

PROOF : It can be easily seen that

$$\frac{2zF'(z)}{F(z) - F(-z)} = \frac{2\left(zf(z) - \int_0^z f(t)\,dt\right)}{\int_0^z f(t)\,dt - \int_0^z f(-t)\,dt}.$$

Let $N = zF'(z)$ and $D = \dfrac{F(z) - F(-z)}{2}$; $D \in S^*$ [since $F(z) \in S_s^*$ Das and Singh 1977b].

Now, $$\frac{N'}{D'} = \frac{2zf'(z)}{f(z) - f(-z)} \text{ and } f \in R(\alpha).$$

Hence by Lemma 1, $F(z) \in R(\alpha)$.

Theorem 5.2 — Let $f(z) \in T(\alpha)$. Then $F(z) = \frac{2}{z} \int_0^z f(t)\, dt$ also belongs to $T(\alpha)$.

This can be proved using Lemma 2.

Theorems 5.1 and 5.2 are analogous to the results of Libera (1965).

6. Representation Formula

Let $P(\alpha)$ denote the class of analytic functions $P(z)$ in E with $p(0) = 1$ and

$$\left| \frac{p(z) - 1}{p(z) + 1} \right| < \alpha,\ 0 < \alpha \leqslant 1,\ z \in E.$$

$P(1)$ coincides with P the class of functions with positive real part.

Theorem 6.1 — Let $f \in R(\alpha)$. Then there exists a function $p(z) \in P(\alpha)$ such that

$$f(z) = \int_0^z p(t) \left\{ \exp\left(\tfrac{1}{2} \int_0^t \frac{p(s) + p(-s) - 2}{s}\, ds \right) \right\} dt \qquad \ldots(6.1.1)$$

and conversely.

Proof : By Lemma A

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 - \alpha w(z)}{1 + \alpha w(z)}.$$

If $p(z) = \frac{1 - \alpha w(z)}{1 + \alpha w(z)}$, then it is clear that $p(z) \in P(\alpha)$.

We have

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z). \qquad \ldots(6.1.2)$$

Changing z to $-z$

$$\frac{2zf'(-z)}{f(z) - f(-z)} = p(-z).$$

Therefore

$$\frac{f'(z)}{f'(-z)} = \frac{p(z)}{p(-z)}$$

$$f'(-z) = p(-z)\, f'(z)/p(z).$$

Also from (6.1.2)

$$-f(-z) = (2zf'(z) - f(z)\,p(z)) \mid p(z).$$

Differentiating, $f'(-z) = \dfrac{2zp(z)\,f''(z) + 2p(z)\,f'(z) - p^2(z)\,f'(z)}{p(z)}.$

Hence $\dfrac{f''(z)}{f'(z)} = \dfrac{p(z) + p(-z) - 2}{2z} + \dfrac{p'(z)}{p(z)}$

$$f(z) = \int_0^z p(t) \left\{\exp\left(\tfrac{1}{2}\int_0^t \frac{p(s) + p(-s) - 2}{s}\, ds\right)\right\} dt.$$

To prove the converse, it is clear that $f(z)$ given by (6.1.1) has the expansion $z + a_2z^2 + \ldots$, and is regular in E. So it is sufficient to show that $f'(z) \neq 0$ and $f \in R(\alpha)$.

We can prove

$$2z \exp\left(\int_0^z \frac{p(s) + p(-s) - 2}{2s}\, ds\right)$$

$$\equiv \int_0^z (p(t) + p(-t)) \left\{\exp\left(\int_0^t \frac{p(s) + p(-s) - 2}{2s}\, ds\right)\right\} dt. \quad \ldots(6.1.3)$$

From (6.1.1) we get

$$f'(z) = p(z) \exp\left(\int_0^z \frac{p(s) + p(-s) - 2}{2s}\, ds\right).$$

Hence $f'(z) \neq 0$ in E.

Also
$$-f(-z) = \int_0^z p(-t) \left\{\exp\left(\int_0^t \frac{p(s) + p(-s) - 2}{2s}\, ds\right)\right\} dt. \quad \ldots(6.1.4)$$

Adding (6.1.1) and (6.1.4) and using (6.1.3) we have

$$(f(z) - f(-z)) = 2z \exp\left(\int_0^z \frac{p(s) + p(-s) - 2}{2s}\, ds\right)$$

$$= 2z\,\frac{f'(z)}{p(z)}$$

or $$\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \quad \text{where} \quad p(z) \in P(\alpha).$$

Therefore $f(z) \in R(\alpha)$.

Theorem 6.2 — Let $Q(\alpha)$ denote the class of analytic functions $q(z)$ in E with $q(0) = 1$ and $| q(z) - \alpha | < \alpha, \alpha > \frac{1}{2}$. Then $f \in T(\alpha)$ iff there exists a function $q(z) \in Q(\alpha)$ such that

$$f(z) = \int_0^z q(t) \left\{\exp\left(\tfrac{1}{2} \int_0^t \frac{q(s) + q(-s) - 2}{s} ds\right)\right\} dt.$$

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