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SOME THEOREMS ON AFFINE MOTION IN A RECURRENT FINSLER SPACE—II

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In the present paper the author studies an affine motion in a recurrent Finsler space and obtains some more results in continuation of his previous paper (Kumar 1977).

1. INTRODUCTION

Let us consider an n-dimensional affinely connected Finsler space Fn (Rund 1959) with a symmetric connection parameter $\Gamma^{*i}_{jk}(x, \dot{x})$. The fundamental metric tensor of the space is given by

$$g_{ij}(x, \dot{x}) \overset{def}{=} \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}). \qquad \ldots(1.1)$$

Let us consider further $T^i_j(x, \dot{x})$ as a mixed tensor depending both upon directional and positional arguments. The covariant derivative of $T^i_j(x, \dot{x})$ with respect to x^k in the sense of Cartan's is given by

$$T^i_{j|k} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_k + T^m_j \Gamma^{*i}_{mk} - T^i_m \Gamma^{*m}_{jk}. \qquad \ldots(1.2)$$

Cartan's connection coefficients $\Gamma^{*i}_{jk}(x, \dot{x})$ satisfy the following relations:

$$\text{(a)} \quad \dot{\partial}_h \Gamma^{*i}_{jk} \dot{x}^h = 0, \quad \text{and} \quad \text{(b)} \quad \dot{\partial}_h \Gamma^{*i}_{jk} = \dot{\partial}_j \Gamma^{*i}_{hk} \cdot \qquad \ldots(1.3)$$

We have the following commutation formula involving Cartan's covariant derivative:

$$2T^i_{j|[hk]} = -\dot{\partial}_r T^i_j K^r_{hk} + T^s_j K^i_{shk} - T^i_s K^s_{jhk} \qquad \ldots(1.4)$$

where

$$K^i_{hjk}(x, \dot{x}) \overset{def}{=} 2\{\partial_{[k}\Gamma^{*i}_{j]h} + \dot{\partial}_s \Gamma^{*i}_{h[j} G^s_{k]} + \Gamma^{*s}_{h[j}\Gamma^{*i}_{k]s}\} \qquad \ldots(1.5)$$

is a curvature tensor which satisfies the following identities: (Rund 1959)

$2A_{[hk]} = A_{hk} - A_{kh}$.

$$K^i_{hjk|s} + K^i_{hks|j} + K^i_{hsj|k} = -\dot{x}^r \{\dot{\partial}_m \Gamma^{*i}_{hj} K^m_{rks} + \dot{\partial}_m \Gamma^i_{hk} K^m_{rsj} + \dot{\partial}_m \Gamma^i_{hs} K^m_{rjk}\} \quad ...(1.6)$$

$$K^i_{hjk} + K^i_{jkh} + K^i_{khj} = 0 \quad ...(1.7)$$

$$\text{(a)} \quad K^i_{hjk} = -K^i_{hkj}, \quad \text{(b)} \quad K_{hj} = K^i_{hji}. \quad ...(1.8)$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt \quad ...(1.9)$$

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant.

Lie-derivatives of $T^i_j(x, \dot{x})$ and $\Gamma^{*i}_{jk}(x, \dot{x})$ are given by

$$\mathcal{L}_v T^i_j(x, \dot{x}) = T^i_{j|h} v^h - T^h_j v^i_{|h} + T^i_h v^h_{|j} + \dot{\partial}_h T^i_j v^h_{|s} \dot{x}^s \quad ...(1.10)$$

and

$$\mathcal{L}_v \Gamma^{*i}_{jk}(x, \dot{x}) = v^i_{|jk} + K^i_{jkh} v^h + \dot{\partial}_r \Gamma^{*i}_{jk} v^r_{|s} \dot{x}^s \quad ...(1.11)$$

respectively (Yano 1957).

We have the following commutation formula:

$$(\mathcal{L}_v \Gamma^{*i}_{jh})_{|k} - (\mathcal{L}_v \Gamma^{*i}_{jk})_{|h} = \mathcal{L}_v K^i_{hjk} + 2\dot{x}^s \dot{\partial}_r \Gamma^{*i}_{h[j} \mathcal{L}_v \Gamma^{*r}_{k]s}. \quad ...(1.12)$$

If Cartan's curvature tensor field $K^i_{hjk}(x, \dot{x})$ of Fn satisfies the relation

$$K^i_{hjk|s} = \gamma_s K^i_{hjk} \quad ...(1.13)$$

where $\gamma(x)$ is a recurrence vector field then the space is called a recurrent Finsler space or Fn^*-space (Kumar 1977).

In view of the infinitesimal point transformation (1.9), we have the following well-known (Wong 1953) theorem.

Theorem 1.1 — In order that (1.9) be an infinitesimal affine motion in an Fn^*, it is necessary and sufficient that Lie-derivative of Γ^{*i}_{jk} with respect to (1.9) vanishes.

$$\mathcal{L}_v \Gamma^{*i}_{jk} = v^i_{|jk} + K^i_{jkh} v^h + \dot{\partial}_r \Gamma^{*i}_{jk} v^r_{|s} \dot{x}^s = 0. \quad ...(1.14)$$

The above result, of course, can be applied to the space Fn^* under consideration. The integrability condition of the last equation is $(\mathcal{L}_v \Gamma^{*i}_{jk})_{|h} - (\mathcal{L}_v \Gamma^{*i}_{jh})_{|k} = 0$

or

$$\mathcal{L}_v K^i_{hjk} = K^i_{hjk|s} v^s - K^s_{hjk} v^i_{|s} + K^i_{sjk} v^s_{|h} + K^i_{hsk} v^s_{|j} + K^i_{hjs} v^s_{|k} + \dot{\partial}_s K^i_{hjk} v^s_{|r} \dot{x}^r = 0 \quad \text{...(1.15)}$$

which by virtue of the recurrence definition (1.13) reduces to

$$\mathcal{L}_v K^i_{hjk} = \gamma_s K^i_{hjk} v^s - K^s_{hjk} v^i_{|s} + K^i_{sjk} v^s_{|h} + K^i_{hsk} v^s_{|j} + K^i_{hjs} v^s_{|k} + \dot{\partial}_h K^i_{hjk} v^s_{|r} \dot{x}^r = 0. \quad \text{...(1.16)}$$

In the following we shall study the possibility of existence of such a motion. Let us consider (1.13) as a partial differential equation with respect to $K^i_{hjk}(x, \dot{x})$, then we can take here a condition

$$0 = (K^i_{hjk|s} - \gamma_s K^i_{hjk})_{|m} - (K^i_{hjk|m} - \gamma_m K^i_{hjk})_{|s}. \quad \text{...(1.17)}$$

In view of eqns. (1.4) and (1.13), the last condition reduces to

$$-(\gamma_{s|m} - \gamma_{m|s}) K^i_{hjk} - \dot{\partial}_r K^i_{hjk} K^r_{lsm} \dot{x}^l + K^r_{hjk} K^i_{rsm} - K^i_{rjk} K^r_{hsm} - K^i_{hrk} K^r_{jsm} - K^i_{hjr} K^r_{ksm} = 0. \quad \text{...(1.18)}$$

If $\gamma_s(x)$ denotes a gradient vector given by $\gamma_{|s}/\gamma$ ($\gamma = \gamma(x)$), eqn. (1.18) yields

$$-\dot{\partial}_r K^i_{hjk} K^r_{lsm} \dot{x}^l + K^r_{hjk} K^i_{rsm} - K^i_{rjk} K^r_{hsm} - K^i_{hrk} K^r_{jsm} - K^i_{hjr} K^r_{ksm} = 0. \quad \text{...(1.19)}$$

From eqns. (1.16) and (1.19) it is clear that if we can take

$$v^i_{|j} = K^i_{jkh} p^{kh} \quad \text{...(1.20)}$$

where p^{kh} is a suitable non-symmetric tensor, then

$$\gamma_s v^s = 0. \quad \text{...(1.21)}$$

The above equation can also be re-written as

$$\mathcal{L}_v \gamma_s(x) = 0. \quad \text{...(1.22)}$$

2. Concurrent Field

Under a concurrent field, we understand a field characterized by (Takano 1961)

$$v^i_{|j} = t\delta^i_j, \quad (t = \text{non-zero const.}). \quad \text{...(2.1)}$$

If there is a motion of the type

$$\bar{x}^i = x^i + v^i(x)\, dt, \quad v^i_{|j} = t\delta^i_j \qquad \text{...(2.2)}$$

we can easily obtain $v^i_{|jk} - v^i_{|kj} = 0$ or

$$v^h K^i_{hjk} = 0 \qquad \text{...(2.3)}$$

where we have used the commutation formula (1.4).

Differentiating (2.3) covariantly in the sense of Cartan and using eqns. (2.2) and (2.3) itself, we get

$$tK^i_{hjk} = 0. \qquad \text{...(2.4)}$$

Since $t \neq 0$, therefore, the space becomes a flat one. Hence there does not exist an affine motion of the form (2.2) in a general Fn^*-space.

3. Special Concircular Field

Let us consider a vector field of the form

$$v^i_{|j} = \psi(x)\, \delta^i_j \qquad \text{...(3.1)}$$

in a special concircular field where $\psi(x)$ means an arbitrary non-zero scalar function. We shall now consider the system of affine motion in a special concircular field of the form:

$$\bar{x}^i = x^i + v^i(x)\, dt, \quad v^i_{|j} = \psi(x)\, \delta^i_j \cdot \qquad \text{...(3.2)}$$

If it will be the case, we can obtain

$$v^i_{|jk} - v^i_{|kj} = \psi_{|k}\delta^i_j - \psi_{|j}\delta^i_k \qquad \text{...(3.3)}$$

which by virtue of the commutation formula (1.4) reduces to

$$v^h K^i_{hjk} = \psi_{|k}\delta^i_j - \psi_{|j}\delta^i_k \cdot \qquad \text{...(3.4)}$$

Differentiating this equation convariantly with respect to x^m and noting the relations (1.13), (3.2) and (3.4) itself, we get

$$\psi K^i_{mjk} + \gamma_m(\psi_{|k}\delta^i_j - \psi_{|j}\delta^i_k) = \psi_{|km}\delta^i_j - \psi_{|jm}\delta^i_k \cdot \qquad \text{...(3.5)}$$

The last equation can also be written like

$$\psi K^i_{mjk} = \delta^i_j\,(\psi_{|km} - \gamma_m\psi_{|k}) - \delta^i_k\,(\psi_{|jm} - \gamma_m\psi_{|j}). \qquad \text{...(3.6)}$$

From the last equation it is clear that for $\psi_{|km} = \gamma_m \psi_{|k}$, we have

$$K^i_{mjk} = 0. \qquad ...(3.7)$$

Thus, in order to avoid a case where the space be reduced to a flat one, we have to assume $\psi_{|km} \neq \gamma_m \psi_{|k}$. That is to say, the gradient vector $\psi_{|k}$ is not a recurrent one with respect to γ_m. Under such an assumption, we consider (3.2) as usual. However, unfortunately if v^i satisfies (3.1), $\psi(x)$ vanishes identically (Takano 1961).

Consequently for $\psi(x) \neq 0$, there is no possibility of the existence of affine motion of the form (3.2) in a general Fn^*.

4. Recurrent Field

If the covariant derivative of the vector field $v^i(x)$ satisfies the relation:

$$v^i_{|j} = \psi_j(x)\, v^i \qquad ...(4.1)$$

where $\psi_j(x)$ denotes an arbitrary covariant vector, the vector field spanned by satisfying (4.1) is called a recurrent field. In the present section we shall study the possibility of an affine motion of the following form:

$$\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{|j} = \psi_j(x)\, v^i \qquad ...(4.2)$$

If an Fn^* admits a motion of the above form, then the vector field v^i has to satisfy (1.14), hence introducing (4.1) into (1.14), we obtain

$$K^i_{jkh}\, v^h = v^i(\psi_{j|k} + \psi_j \psi_k) \qquad ...(4.3)$$

where we have used (4.1) in the process of calculation. Transvecting the above equation by v^k and using the skew symmetric property of the curvature tensor K^i_{jkh} into the indices k and h, we get

$$\psi_{j|k} v^k v^i + \psi_j \psi_k v^k v^i = 0 \qquad ...(4.4)$$

from which follows that

$$\psi_{j|k} v^k + \psi_j \psi_k v^k = 0. \qquad ...(4.5)$$

Next, differentiating (4.3) covariantly with respect to x^m and using eqns. (1.13) and (4.1), we have

$$\psi_{j|km} v^i + \psi_{j|k} \psi_m v^i + \psi_{j|m} \psi_k v^i + \psi_j \psi_{k|m} v^i + \psi_j \psi_k \psi_m v^i$$
$$= (\gamma_m + \psi_m)\, K^i_{jkh}\, v^h. \qquad ...(4.6)$$

Contracting the above equation with respect to the indices i and k we get

$$\psi_{j|km}v^k + \psi_{j|k}\psi_m v^k + \psi_{j|m}\psi_k v^k + \psi_j\psi_{k|m}v^k + \psi_j\psi_k v^k\psi_m = -(\gamma_m + \psi_m)\, K_{jh}v^h \qquad ...(4.7)$$

where we have used eqn. (1.8b).

Again, differentiating (4.5) covariantly with respect to x^m and remembering eqn. (4.1), we obtain

$$\psi_{j|km}v^k + \psi_{j|k}\psi_m v^k + \psi_{j|m}\psi_k v^k + \psi_j\psi_{k|m}v^k + \psi_j\psi_k\psi_m v^k = 0. \qquad ...(4.8)$$

Introducing the left-hand side of the above equation into the left-hand side of (4.7), we obtain

$$(\gamma_m + \psi_m)\, K_{jh}v^h = 0 \qquad ...(4.9)$$

from which, we get

$$\psi_m = -\gamma_m \text{ or } K_{jh}v^h = 0\dagger. \qquad ...(4.10)$$

For the first case the system of motion can be written as

$$\bar{x}^i = x^i + v^i(x)\, dt,\; v^i_{|j} = -\gamma_j v^i. \qquad ...(4.11)$$

In the following lines we shall seek to obtain a necessary and sufficient condition for the existence of a motion of the form (4.11). In order that (4.11) construct an affine motion it is necessary and sufficient that (4.11) satisfy (1.16). In order to obtain an essential condition for our purpose, let us substitute the value of $v^i_{|j}$ from (4.11) into eqn. (1.16), we obtain

$$\begin{aligned}\mathcal{L}_v K^i_{hjk}(x, \dot{x}) &= \gamma_s v^s K^i_{hjk} + \gamma_s K^s_{hjk}\, v^i - \gamma_h K^i_{sjk}\, v^s - \gamma_j K^i_{hsk}\, v^s - \gamma_k K^i_{hjs}\, v^s \\ &= v^s(\gamma_s K^i_{hjk} + \gamma_j K^i_{hsk} - \gamma_k K^i_{hjs}) + \gamma_s v^i K^s_{hjk} - \gamma_h v^s K^i_{sjk} \\ &= v^s(K^i_{hjk|s} + K^i_{hks|j} + K^i_{hsj|k}) - K^s_{hjk}\, v^i_{|s} + K^i_{sjk}\, v^s_{|h}\end{aligned} \qquad ...(4.12)$$

where we have used eqns. (1.8a), and (4.11).

In an affinely connected Finsler space the identity (1.6) reduces to

$$K^i_{hjk|s} + K^i_{hks|j} + K^i_{hsj|k} = 0. \qquad ...(4.13)$$

Hence, by virtue of the above identity the eqn. (4.12) yields

$$\mathcal{L}_v K^i_{hjk} = -K^s_{hjk}\, v^i_{|s} + K^i_{sjk}\, v^s_{|h} \qquad ...(4.14)$$

†We shall consider afterward the case where $K_{jh}v^h = 0$ (see Kumar 1977).

On the other hand, we have generally

$$v^i_{|s} K^s_{hjk} - v^s_{|h} K^i_{sjk} = v^i_{|hjk} - v^i_{|hkj}. \quad ...(4.15)$$

Therefore, in order to get $\mathcal{L}_v K^i_{hjk} = 0$, in the present case it is necessary and sufficient that

$$v^i_{|hjk} - v^i_{|hkj} = 0 \quad ...(4.16)$$

and in this case only (4.11) becomes an affine motion of the space. The condition (4.16) is an integrability condition of the equation

$$(\sigma(x)\, v^i_{|h})_{|j} = 0 \quad ...(4.17)$$

where $\sigma(x)$, a non-zero arbitrary function, depends only upon the positional coordinates. Consequently, in order to get the affine motion (4.11), it is necessary and sufficient that (4.17) be assumed. Substituting the eqn. (4.11) in eqn. (4.17), we have,

$$(-\sigma_{|j}\gamma_h - \sigma\gamma_{h|j} + \sigma\gamma_h\gamma_j)\, v^i = 0. \quad ...(4.18)$$

So for a non-zero vector field v^i the above equation yields

$$\gamma_{h|j} = \gamma_h\gamma_j - \sigma_j\gamma_h \quad ...(4.19)$$

where

$$\sigma_j \equiv \sigma_{|j}/\sigma. \quad ...(4.20)$$

Therefore, we can now consider the motion of the form

$$\bar{x}^i = x^i + v^i(x)\, dt, \ v^i_{|j} = -\gamma_j v^i, \ \gamma_{h|j} = \gamma_h\gamma_j - \sigma_j\gamma_h. \quad ...(4.21)$$

Conversely, if we have (4.21), from this equation, we can obtain the original form of (4.17).

If we are able to consider (4.16) excepting for a scalar proportionality $\sigma_j(x)$, that $\sigma_j = 0$ in place of eqn. (4.21), we get

$$\bar{x}^i = x^i + v^i(x)\, dt, \ v^i_{|j} = -\gamma_j v^i, \ \gamma_{h|j} = \gamma_h\gamma_j. \quad ...(4.22)$$

The last of the above equations shows that γ_h be a gradient covariant vector:

$$\gamma_j = \gamma_{|j}/\gamma, \ \gamma = \gamma(x). \quad ...(4.23)$$

In this case, we have $(v^s\gamma_s)_{|j} = (\gamma_{s|j} - \gamma_s\gamma_j) = 0$, i.e. $v^s\gamma_s = \text{const}$. Taking this constant to be zero, we have $v^s\gamma_s = 0$, i.e.

$$\mathcal{L}_v\gamma_s(x) = 0.$$

Thus, we have the following:

Theorem 4.1 — An Fn^*-space is able to admit an affine motion of the recurrent form:

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{\mid j} + \gamma_j v^i$$

with an additional relation (4.19) being assumed to be integrable.

Corollary 4.1 — The Fn^*-space is able to have an affine motion of contra-form:

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{\mid j} + \gamma_j v^i = 0\dagger$$

with the condition:

(i) $\gamma_{j \mid h} = \gamma_j \gamma_h$, (ii) $\mathcal{L}_v \gamma_s(x) = 0$

where $\gamma_j = \gamma_{\mid j}/\gamma$.

Such a motion has been introduced under the solvability of the characteristic eqn. (4.19). However, eqn. (4.19) has actually a special solution $\gamma_h = 0$. Hence, the following existence theorem of a contra-motion holds good.

Theorem 4.2 — The space Fn^* is able to have naturally an affine motion of contra-form in the strict sense: $\bar{x}^i = x^i + v^i(x)$, $v^i_{\mid j} = 0$. For $\gamma_j = 0$, eqn. (1.13) reduces to

$$K^i_{hjk \mid s} = 0 \qquad \ldots(4.24)$$

and this is the condition for symmetric Finsler space. Thus, we have the following

Theorem 4.3 — In order that an Fn^* admitting an affine motion of the recurrent form of the type (4.21) be a symmetric space, it is necessary and sufficient that the motion be taken to be a contra-form in the strict sense.

5. Some Essential Conditions

Let us consider the characteristic differential eqn. (4.19) of the defining covariant vector for the recurrent affine motion. The integrability condition of (4.19) is given by

$$(\gamma_{h \mid j} - \gamma_h \gamma_j + \sigma_j \gamma_h)_{\mid k} - (\gamma_{h \mid k} - \gamma_h \gamma_k + \sigma_k \gamma_h)_{\mid j} = 0. \qquad \ldots(5.1)$$

By virtue of the commutation formula (1.4), the above equation yields

$$\gamma_s K^s_{hjk} = (\gamma_{j \mid k} - \gamma_{k \mid j})\, \gamma_h \qquad \ldots(5.2)$$

†According to Wong (1953), under a contra-field, we mean a field of parallel contravariant vectors $v^i_{\mid j} + \gamma_j v^i = 0$ can also be written like $(\gamma v^i)_{\mid j}$, so v^i spans a contra-field except for a scalar proportionality. In case of $\gamma_j =$ const., we shall call such a field a contra-field *in the strict sense*.

i.e.

$$\gamma_s K^s_{hjk} = (-\sigma_k\gamma_j + \sigma_j\gamma_k)\,\gamma_h. \qquad ...(5.3)$$

If we take the arbitrary covariant vector σ_k as a gradient vector being equal to zero or γ_k, the last two equations reduce to

$$\gamma_s K^s_{hjk} = 0. \qquad ...(5.4)$$

But, in view of this equation the integrability condition (4.19) reduces to

$$\gamma_{h\,|\,j} = \gamma_h\gamma_j \quad \text{or} \quad \gamma_{h\,|\,j} = 0 \qquad ...(5.5)$$

and (5.4) holds identically.

Equation (4.21) of motion is equivalent to the system of the relations (4.11) and (4.17), i.e. to that of (4.11) and

$$\sigma_j v^i_{\,|\,h} + v^i_{\,|\,hj} = 0. \qquad ...(5.6)$$

Substituting the value of the term $v^i_{\,|\,hj}$ from (1.14) in the above equation, we obtain

$$\sigma_j v^i_{\,|\,h} = -K^i_{hjs}\, v^s + \dot{\partial}_s \Gamma^{*i}_{hj}\, v^s_{\,|\,r}\, \dot{x}^r \qquad ...(5.6a)$$

which in view of eqn. (4.11) reduces to

$$\gamma_h \sigma_j v^i = K^i_{hjs}\, v^s. \qquad ...(5.7)$$

Transvecting the last relation by v^j and using $K^i_{hjs}\, v^j v^s = 0$, we get

$$\sigma_j \gamma_h v^i v^j = 0. \qquad ...(5.8)$$

But for a non-zero v^i and γ_h, the above relation yields

$$\sigma_j v^j = 0 \quad \text{or} \quad \mathcal{L}_v \sigma(x) = 0. \qquad ...(5.9)$$

That is the scalar function $\sigma(x)$ is a Lie-invariant one.

Contracting eqn. (5.7) with respect to the indices i and j, we obtain

$$\gamma_h \sigma_j v^j = K^i_{his}\, v^s = -K^i_{hs}\, v^s \qquad ...(5.10)$$

where we have used the eqns. (1.8) and (5.9).

By virtue of eqn. (1.8a) contracting the identity (4.13) with respect to the indices i and s, we have

$$K^s_{hjk\,|\,s} = K^s_{hjs\,|\,k} - K^s_{hks\,|\,j} \qquad ...(5.11)$$

and which by virtue of (1.8*b*) and (1.13) reduces to

$$\gamma_s K^s_{hjk} = \gamma_k K_{hj} - \gamma_j K_{hk}. \quad \text{...(5.12)}$$

Equating the left-hand sides of the two equations, (5.3) and (5.12), we get

$$\gamma_k(K_{hj} - \gamma_h \sigma_j) - \gamma_j(K_{hk} - \gamma_h \sigma_k) = 0. \quad \text{...(5.13)}$$

Multiplying the above equation by v^k and summing over k, we obtain

$$\gamma_k v^k (K_{hj} - \gamma_h \sigma_j) = 0 \quad \text{...(5.14)}$$

where we have used (5.9) and (5.10). Hence if an Fn^* admits an affine motion of the recurrent form (4.21), we have

$$K_{hj} = \gamma_h \sigma_j \quad \text{or} \quad \gamma_k v^k = 0. \quad \text{...(5.15)}$$

We shall now study the integrability condition of $v^i_{|j} + \gamma_j v^i = 0$. From

$$(v^i_{|j} + \gamma_j v^i)_{|k} - (v^i_{|k} + \gamma_k v^i)_{|j} = 0$$

we get

$$v^h K^i_{hjk} = v^i(-\gamma_{j|k} + \gamma_{k|j}) \quad \text{...(5.16)}$$

where we have used the commutation formula (1.4) and (4.21). Transvecting the identity (1.7) by v^h and using (1.8*a*), we have

$$K^i_{hjk}\, v^h = -K^i_{jkh}\, v^h + K^i_{kjh}\, v^h. \quad \text{...(5.17)}$$

Introducing (5.7) into the right-hand side of the above equation, we get (5.16), i.e. the integrability condition (5.16) holds identically.

6. Discussion on the Vector

We have assumed the existence of a gradient vector σ_k derived from an arbitrary scalar function $\sigma(x)$ satisfying (4.17), and in the course of discussion of the integrability conditions of (4.19), we have taken up the following two cases:

$$(a) \quad \sigma_k = 0 \quad \text{and} \quad (b) \quad \sigma_k = \gamma_k \quad \text{...(6.1)}$$

but it remains to be proved whether these are possible or not.

Differentiating covariantly eqn. (1.14) and using the latter part of (4.17), we obtain

$$v^i_{|jks} = -\gamma_s K^i_{jkh}\, v^h + \gamma_s K^i_{jkh}\, v^h = 0. \quad \text{...(6.2)}$$

Again taking the covariant derivative of (5.6) with respect to x^s and using the eqns. (6.2) and (5.6) itself, we get

$$\sigma_{j|s} v^i_{|h} + \sigma_j v^i_{|hs} = 0 \qquad \text{...(6.3)}$$

that is

$$\sigma_{j|s}(-\gamma_h v^i) + \sigma_j(-\sigma_s v^i_{|h}) = 0$$

or

$$\sigma_{j|s}(-\gamma_h v^i) + \sigma_j \sigma_s \gamma_h v^i = 0. \qquad \text{...(6.4)}$$

Consequently, if an Fn^*-space admits an affine motion of the present form (4.21) for a non-zero v^i and γ_h, we obtain

$$\sigma_{j|s} - \sigma_j \sigma_s = 0. \qquad \text{...(6.5)}$$

That is the characteristic equation of $\sigma(x)$ introduced by (4.17) *a priori.* The integrability condition of the above equation is given by

$$\sigma_h K^h_{jsm} = 0. \qquad \text{...(6.6)}$$

In this way, we have to assume (6.5) and (6.6) in our discussion of affine motion. Thus, at this moment we can discuss three cases (6.1*a*), (6.1*b*) and (5.15). At first (6.5) has a solution $\sigma_j = 0$ and therefore the first case (6.1*a*) is always possible.

Secondly, if $\sigma_k = \gamma_k$, the eqn. (6.5) becomes

$$\gamma_{j|s} = \gamma_j \gamma_s. \qquad \text{...(6.7)}$$

By virtue of the last equation the relation (4.19) yields

$$\gamma_{h|j} = 0 \qquad \text{...(6.8)}$$

and therefore in the light of this eqn. (6.7) reduces to

$$\gamma_j = 0. \qquad \text{...(6.8a)}$$

And for this condition the space becomes a symmetric one. Therefore a non-symmetric Fn^*-space cannot admit the case (6.1*b*) and when it is the case Fn^* must be a symmetric space. Thus, we can easily see that for $\sigma_k = 0$ the space Fn^* can admit a contra-motion but the case of $\gamma_k = \sigma_k$ is unable to consider in a general Fn^*-space. Accordingly, if we treat only a general non-symmetric space, we have to regard σ_k as a vector being not equal to γ_k.

Now we shall consider the condition (5.15). First of all we shall discuss the latter:

$$\gamma_h v^h = 0, \ (\gamma_h \neq 0). \qquad \text{...(6.9)}$$

Since, we have always $\mathcal{L}_v\sigma(x) = 0$ or $\sigma_h v^h = 0$, we may consider a special case such that

$$\sigma_h = \delta\gamma_h \qquad ...(6.10)$$

where $\delta = \delta(x)$ non-zero and non-constant scalar function. By virtue of the above equation the relation (4.19) yields

$$\gamma_{h\,|\,j} = (1 - \delta)\,\gamma_h\gamma_j. \qquad ...(6.11)$$

In view of the above equation, we get

$$\gamma_{h\,|\,j} = \gamma_{j\,|\,h} \qquad ...(6.12)$$

which shows that γ_h is a gradient vector. Then, let us consider the concrete form of γ_h. In this case the integrability condition of (4.19) becomes

$$\gamma_s K^s_{hjk} = 0. \qquad ...(6.13)$$

On the other hand, the deformed equation of (4.19) shown above gives us its integrability condition of the form

$$\gamma_s K^s_{hjk} = (-\,\delta_{\,|\,k}\gamma_j + \gamma_k\delta_{\,|\,j})\,\gamma_h. \qquad ...(6.14)$$

Consequently, γ_j must satisfy

$$\gamma_j = \epsilon\,\delta_{\,|\,j}\ (\epsilon = \epsilon(x)):\ \text{suitable function.} \qquad ...(6.15)$$

Introducing eqn. (6.10) into the left-hand side of (6.5) and using the relation (4.19), we obtain

$$(2\delta - 1)\,\delta\gamma_s = \delta_{\,|\,s}. \qquad ...(6.16)$$

Therefore, we can get

$$\delta = \frac{1}{(2\delta - 1)\,\delta}. \qquad ...(6.17)$$

In view of the last equation, we can also obtain

$$\gamma_j = \frac{\delta_{\,|\,j}}{\delta(2\delta - 1)} = -\frac{1}{\delta}\,\delta_{\,|\,j} + \frac{2}{2\delta - 1}\,\gamma_{\,|\,j} \qquad ...(6.18)$$

that is, γ_j denotes certainly a gradient. Furthermore

$$\gamma_j = \frac{1}{\delta}\cdot\frac{1}{\sigma}\,\sigma_{\,|\,j}. \qquad ...(6.19)$$

Equating the eqns. (6.18) and (6.19), we have

$$\frac{1}{2\delta - 1}\,\delta_{\,|\,j} = \frac{1}{\sigma}\,\sigma_{\,|\,j}. \qquad ...(6.20)$$

We can find specially at last

$$\delta = \tfrac{1}{2}(1 + \sigma^2). \quad \text{...(6.21)}$$

7. The Condition $K_{jh}v^h = 0$ and Integrability of $K_{jh} = \gamma_j\sigma_h$

In §4, in order to find the motion of the form

$$\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{\ |j} = \psi_j v^i \quad \text{...(7.1)}$$

we have obtained the conditions

$$\psi_j = -\gamma_j \quad \text{or} \quad K_{jh}v^h = 0. \quad \text{...(7.2)}$$

Here the latter condition has been accepted from our study. However, with the help of the condition (5.15) it is clear that from the former condition follows always the latter condition. Hence the first condition is a special case of the second. Now we shall consider only the form

$$K_{jh}v^h = 0. \quad \text{...(7.3)}$$

In this case, we shall show that we can associate naturally this exceptional case itself with our present theory. In fact, if (7.3) be the case, differentiating this covariantly in the sense of Cartan, we get

$$(\gamma_m + \psi_m)\, K_{jh}v^h = 0. \quad \text{...(7.4)}$$

From the above equation it is clear that the quantity within the bracket (i.e. $\gamma_m + \psi_m$) can be taken to be quite arbitrary. So, we can put

$$\psi_m + \gamma_m = 0 \quad \text{or} \quad \psi_m = -\gamma_m. \quad \text{...(7.5)}$$

Thus, the recurrent condition $v^i_{\ |j} = \psi_j v^i$ becomes $v^i_{\ |j} + \gamma_j v^i = 0$. In this way, we can associate the condition (7.3) with our standpoint. Now the integrability condition of

$$K_{jh} = \gamma_j\sigma_h \quad \text{or} \quad \sigma K_{jh} = \gamma_j\sigma_{|h} \quad \text{...(7.6)}$$

will be calculated and proved with ease. For this, differentiating the latter part of (7.6) covariantly with respect to x^m, we get

$$\gamma_{j|m}\sigma_{|h} + \gamma_j\sigma_{|hm} - \sigma_{|m}K_{jh} - \sigma K_{jh|m} = 0. \quad \text{...(7.7)}$$

In view of eqns. (6.5) and (7.6) the above equation reduces to

$$\gamma_{j|m}\sigma_{|h} + \gamma_j(\sigma_{|m}\sigma_h + \sigma\sigma_h\sigma_m) - \gamma_j\sigma_h\sigma_{|m} - \sigma\gamma_m\gamma_j\sigma_h = 0. \quad \text{...(7.8)}$$

Interchanging the indices h and m in the above equation, we get a similar relation. Subtracting the equation thus obtained from (7.8), we get

$$(\gamma_{j|m} - \gamma_j\gamma_m + \sigma_m\gamma_j)\,\sigma_{|h} - (\gamma_{j|h} - \gamma_j\gamma_h + \sigma_h\gamma_j)\,\sigma_{|m} = 0. \quad \text{...(7.9)}$$

But, in order to get the present motion, we have assumed (4.19), hence above condition holds identically. That is (7.6) is completely integrable.

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