Indian J. pure appl. Math., 11(4) : 407-412, April 1980

LAPLACE TRANSFORM SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS

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(*Received* 26 *March* 1979; *after revision* 11 *December* 1979)

A technique for the solution of linear differential equations is extended to find the solution of a class of nonlinear differential equations. The nonlinear equation is first transformed from the time domain to the multidimensional frequency domain by introducing the concept of nonlinear transfer function. Transformation from multidimensions to single dimension is obtained by the technique of association of variables. Time domain solution of the equation is then found by inverse Laplace transform.

INTRODUCTION

This paper considers a nonlinear differential equation of the form

$$g\left(\frac{d}{dt}\right)y + \sum_{i=1}^{n} a_i y^i = x(t). \qquad \text{...(1)}$$

All the initial conditions are zeros.

Equation (1) can be written as

$$f(y(t), \dot{y}(t), \ddot{y}(t) \ldots) = x(t)$$

or $$K[y(t)] = x(t)$$

or $$y(t) = H[x(t)]$$

K is a nonlinear operator (Waddington and Fallside 1966) and H is the inverse of K.

ANALYSIS OF A NONLINEAR SYSTEM

Equation (1) can be viewed as follows:

$x(t)$ is the input to a nonlinear system and $y(t)$ is the output of the system as shown in Fig. 1. Volterra (1959) has shown that the output $y(t)$ of the nonlinear system is some functional of its input $x(t)$. The following functional expansion was suggested

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$$y(t) = \int_{-\infty}^{\infty} h_1(\tau_1)\, x(t - \tau_1)\, d\tau_1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)\, x(t - \tau_1)\, x(t - \tau_2)\, d\tau_1 d\tau_2$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tau_1, \tau_2, \tau_3)\, x(t - \tau_1)\, x(t - \tau_2)\, x(t - \tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 + \ldots \quad \ldots(2)$$

$h_1(\tau_1)$ is the impulse response of a linear system, $h_2(\tau_1, \tau_2)$ is the impulse response of a quadratic system and $h_n(\tau_1, \tau_2, \ldots, \tau_n)$ is the generalized impulse response of an nth order system. Let

$$y_n(t) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n)\, x(t - \tau_1)\, x(t - \tau_n) \ldots x(t - \tau_n)\, d\tau_1\, d\tau_2 \ldots d\tau_n.$$

Then eqn. (2) can be written in the form

$$y(t) = \sum_{n=1}^{\infty} y_n(t). \quad \ldots(3)$$

Hence, the arbitrary nonlinear system of Fig. 1 can be represented by a sequence of subsystems connected in parallel as shown in Fig. 2.

Depending upon the nature of the nonlinearity and the magnitude of the input, the series represented by eqn. (3) will be either convergent or divergent. In the event the series is convergent, it will be sufficient to represent $y(t)$ by the first few nonzero terms of the functional expansion (2). The necessary condition that the functional expansion will be convergent is that both $\int_{-\infty}^{\infty} | h_1(t) |\, dt$ and $x(t)$ be bounded. Physically, $y_1(t)$ is the response of a first order or linear system and $y_n(t)$ is the response of the nth order system.

Frequency Domain Analysis

From the linear system theory (Schwarz and Friedland 1965) the output of the first order system is simply a convolution of the input $x(t)$ and the impulse response $h_1(t)$. In the frequency domain it is given by

$$Y_1(s_1) = H_1(s_1)\, X(s_1). \quad \ldots(4)$$

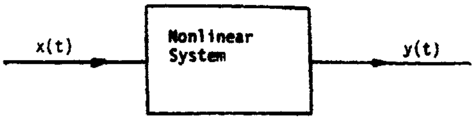

FIG. 1. Input-output relationship of a nonlinear system.

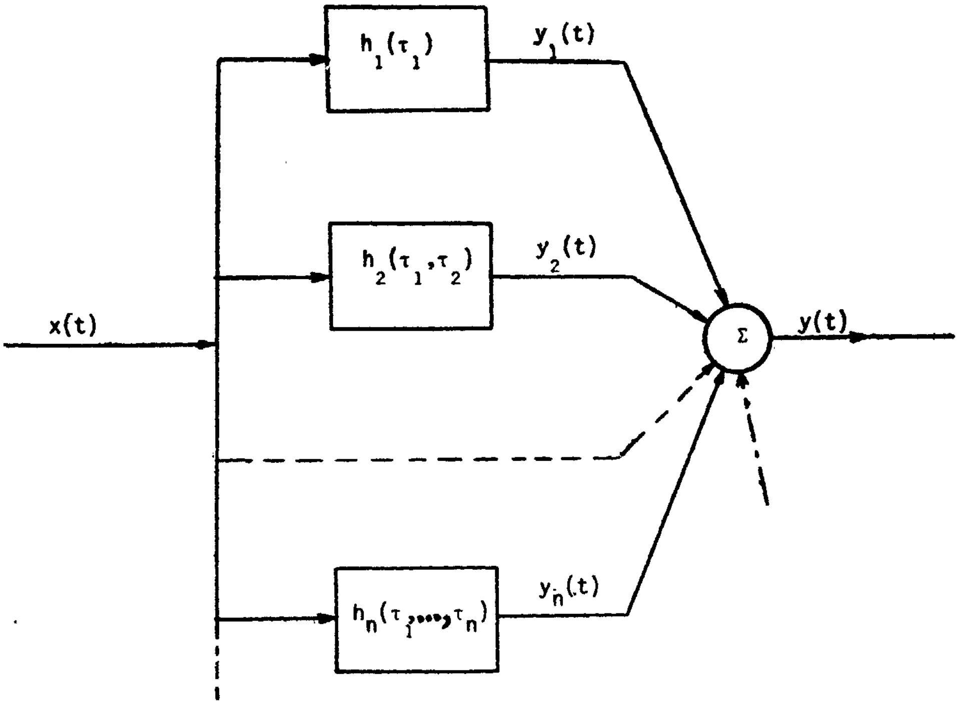

FIG. 2. Functional expansion of the nonlinear system in Fig. 1.

The output of the second order system is given by

$$y_2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)\, x(t_1 - \tau_1)\, x(t - \tau_2)\, d\tau_1\, d\tau_2. \qquad \ldots(5)$$

By letting $t_1 = t_2 = t$, eqn. (5) can be written as

$$y_2(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)\, x(t_1 - \tau_1)\, x(t_2 - \tau_2)\, d\tau_1\, d\tau_2.$$

By taking two dimensional transform we get from above

$$Y_2(s_1, s_2) = H_2(s_1, s_2)\, X(s_1)\, X(s_2).$$

Similarly for the third order system

$$Y_3(s_1, s_2, s_3) = H_3(s_1, s_2, s_3)\, X(s_1)\, X(s_2)\, X(s_3).$$

In general

$$Y_n(s_1, s_2, \ldots, s_n) = H_n(s_1, s_2, \ldots, s_n) \prod_{i=1}^{n} X(s_i). \qquad \ldots(6)$$

Now, if $H_n(s_1, s_2, \ldots, s_n)$ is known, the output spectrum $Y_n(s_1, s_2, \ldots, s_n)$ can be expressed in terms of the input spectrum $X(s_1), X(s_2), \ldots, X(s_n)$. By inverting $Y_n(s_1, s_2, \ldots, s_n)$ and by letting $t = t_1 = t_2, \ldots, t_n$, $y_n(t)$ can be obtained for some given $x(t)$.

Computation of Nonlinear Transfer Function

In the mathematical literature the quantity $H_n(s_1, s_2, \ldots, s_n)$ is interchangeably termed as Volterra kernel, multidimensional transform or nonlinear transfer function. As an analogy to the linear system, the nonlinear transfer function (Kuo 1977) is defined to be

$$H_n(s_1, s_2, \ldots, s_n)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) \exp[-(s_1\tau_1 + \ldots + s_n\tau_n)]\, d\tau_1 \ldots d\tau_n. \qquad \ldots(7)$$

One method of determining the Volterra kernel is harmonic input method (Kuo 1977). This method is based upon the fact that a harmonic input to the system produces harmonic output. When the system in Fig. 1 is excited with a set of n unit amplitude exponentials at the noncommensurate frequencies $s_1, s_2, \ldots, s_n$, the output of the nonlinear system is given by

$$y(t) = \sum_{n=1}^{\infty} \sum_{m} \frac{n!}{m_1!\, m_2! \ldots m_n!} H_n(s_1, s_2, \ldots, s_n) \exp(s_1 + s_2 + \ldots + s_n)\, t. \qquad \ldots(8)$$

While m under the summation indicates that for each n the sum is to be taken over all distinct values of m_n, $1 \leqslant m_1 \leqslant m_2 \ldots \leqslant m_n$ such that $m_1 + m_2 + \ldots + m_n = n$. The harmonic input method thus states that the nth order nonlinear transfer function can be obtained analytically as the coefficient of $n! \exp(s_1 + s_2 + \ldots + s_n)\, t$ in the system output when the input to the system is

$$x(t) = \exp(s_1 t) + \exp(s_2 t) + \ldots + \exp(s_n t). \qquad \ldots(9)$$

Since the coefficient of $\exp(s_1 + s_2 + \ldots + s_n)\, t$ is given by $n!\, H_n(s_1, s_2, \ldots, s_n)$ when the input is given by (9), it is easy to see that $H_n(s_1, s_2, \ldots, s_n)$ can be expressed in terms of $H_{n-1}(s_1, s_2, \ldots, s_{n-1})$.

Example of the Solution of Nonlinear Differential Equation

We shall clarify our above discussion with the help of an example. We consider the following nonlinear differential equation:

$$\dot{y} + \alpha y^2 = \beta t^n = x(t) \qquad \ldots(10)$$

$\alpha, \beta \neq 0$, n is a positive integer.

To find the first order transfer function $H_1(s_1)$ we let $x(t) = \exp(s_1 t)$ and $y(t) = H_1(s_1)\exp(s_1 t)$. Substituting the above values of $x(t)$ and $y(t)$ in (10) and equating the coefficient of $\exp(s_1 t)$ on both sides we obtain

$$H_1(s_1) = 1/s_1.$$

To determine the second order transfer function $H_2(s_1, s_2)$, we let

$$x(t) = \exp(s_1 t) + \exp(s_2 t)$$

and

$$y(t) = H_1(s_1)\exp(s_1 t) + H_1(s_2)\exp(s_2 t) + 2H_2(s_1, s_2)\exp(s_1 + s_2)\,t.$$

Substituting the above values of $x(t)$ and $y(t)$ in (10) and equating the coefficient of $2!\exp(s_1 + s_2)\,t$ we obtain

$$H_2(s_1, s_2) = -\alpha H_1(s_1)\,H_1(s_2)\,H_1(s_1 + s_2).$$

Similarly $H_3(s_1, s_2, s_3)$ can be determined.

In solving (10) we notice that the input $x(t)$ is not bounded and even though the nonlinearity is only second order, the functional expansion will have an infinite number of terms (Bussang *et al.* 1974) and will be divergent. However, to obtain the first two nonzero terms, from (6) we get

$$Y(s_1, s_2) = H_1(s_1)\,X(s_1) + H_2(s_1, s_2)\,X(s_1)\,X(s_2).$$

Now the right-hand side of the equation can be written as

$$\frac{\beta n!}{s_1^{n+2}} + \frac{\alpha\beta^2(n!)^2}{(s_1 + s_2)\,s_1^{n+2}\cdot s_2^{n+2}}.$$

Now by the technique called association of variables (Lubbock and Bansal 1969), we intend to translate the above quantity into one frequency domain. The first quantity $\beta n!/s_1^{n+2}$ is translated into $\beta n!/s^{n+2}$ since it contains only one frequency s_1. To translate the second quantity into one frequency domain we write it in the form

$$\alpha\beta^2(n!)^2\,H(s_1 + s_2)\,F_1(s_1, s_2)$$

where $H(s_1 + s_2) = \dfrac{1}{s_1 + s_2}$ and $F_1(s_1, s_2) = \dfrac{1}{s_1^{n+2}}\cdot\dfrac{1}{s_2^{n+2}}$.

By using the table for associated transform (Lubbock and Bansal 1969), we get

$$H(s) = \frac{1}{s} \text{ and } F_1(s) = \frac{(2n+2)!}{(n+1)!\,(n+1)!\,s^{2n+3}}.$$

We now make use of the real convolution theorem (Chen and Chiu 1973) which states that if $F(s_1, s_2, \ldots, s_n) = H(s_1 + s_2 + \ldots + s_n)\, F_1(s_1, s_2, \ldots, s_n)$ and if $F(s)$, $H(s)$ and $F_1(s)$ are their respective associated transforms, then

$$F(s) = H(s)\, F_1(s).$$

Using the above results finally we get

$$Y(s) = \frac{\beta n!}{s^{n+2}} - \frac{\alpha\beta^2}{(n+1)^2\,(2n+3)} \cdot \frac{(2n+3)!}{s^{2n+4}}.$$

By taking the inverse Laplace transform (Hodgman 1959)

$$y(t) = \frac{\beta t^{n+1}}{n+1} - \frac{\alpha\beta^2 t^{2n+3}}{(n+1)^2\,(2n+3)}.$$

In order to check the accuracy of the above method let $n = 1$, $\alpha = -1$ and $\beta = 1$. Thus

$$y(t) = \frac{t^2}{2} + \frac{t^5}{20}$$

which has the same first two nonzero terms obtained by the numerical solution of the equation $\dot{y} - y^2 = t$.

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