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SOME FIXED POINT THEOREMS IN METRIC AND BANACH SPACE

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Several fixed point theorems in metric and Banach space have been obtained for mappings satisfying a new contraction type condition. It has been shown that in a convex complete metric space our mapping becomes a Banach contraction mapping.

§1. The present paper is a sequel to the author's previous work (Khan 1976, 1978). In Khan (1976), the following fixed point theorem was proved.

Theorem — Let T be a self mapping of a complete metric space (X, d) and satisfying

(*) $$d(Tx, Ty) \leqslant \alpha \{d(x, Tx)\, d(y, Ty)\}^{1/2}$$

for all $x, y \in X$ and $0 \leqslant \alpha < 1$. Then T has a unique fixed point.

Several results on fixed points for such a mapping were obtained in Khan (1978). In this paper, we continue our study of fixed points for mappings satisfying (*) and a similar condition, namely

(**) $$d(Tx, Ty) \leqslant \alpha \{d(x, Ty)\, d(y, Tx)\}^{1/2}$$

in metric and Banach space.

§2. In this section fixed point theorems in a metric space have been obtained.

Theorem 1 — Let T_1 and T_2 be two continuous mappings of a complete metric space (X, d) into itself such that

$$d(T_1x, T_2y) < d(x, y),\ x \neq y \in X$$

and there exists a subset $A \subset X$ and a point $x_0 \in A$ satisfying the following conditions:

(i) $$d(x_0, T_ix) - d(T_1x_0, T_1T_2x) \geqslant 2d(x_0, T_1x_0) \text{ for } x, y \text{ in } X - A,\ i = 1, 2$$

and $T_1T_2 = T_2T_1$,

(ii) $$d(T_1x, T_2y) \leqslant \alpha(d(x, y))\, \{d(x, T_1x)\, d(y, T_2y)\}^{1/2} \text{ for } x, y \text{ in } A,$$

where α is a monotonically decreasing function from $[0, \infty)$ into $[0, 1)$. Then there exists a unique common fixed point of T_1 and T_2.

PROOF : We suppose that $x_0 \neq T_1 x_0$ and define a sequence $\{x_n\}$ of elements $x_n \in X$,

$$T_1 x_0 = x_1,\ T_2 x_1 = x_2,\ \ldots,\ T_1 x_{2n} = x_{2n+1},\ T_2 x_{2n+1} = x_{2(n+1)}.$$

From $d(T_1 x, T_2 y) < d(x, y)$, the sequence $d(x_n, x_{n+1})$ is non-increasing and $d(x_n, x_{n+1}) < d(x_0, x_1)$, for $n = 1, 2, 3, \ldots$.

Now from the triangle inequality and $T_1 T_2 = T_2 T_1$,

$$\begin{aligned} d(x_0, x_{2n+1}) &\leqslant d(x_0, x_1) + d(x_1, x_{2n+2}) + d(x_{2n+2}, x_{2n+1}) \\ &= d(x_0, x_1) + d(T_1 x_0, T_2 T_1 x_{2n}) + d(x_{2n+2}, x_{2n+1}) \\ &< 2d(x_0, x_1) + d(T_1 x_0, T_2 T_1 x_{2n}). \end{aligned}$$

Thus

$$d(x_0, T_1 x_{2n}) - d(T_1 x_0, T_2 T_1 x_{2n}) < 2d(x_0, T_1 x_0).$$

Hence from the condition (i), it follows that $x_{2n+1} \in A$ for every n. Similarly $x_{2n+2} \in A$ for all n. Therefore $x_n \in A$ for each n.

Next we show that the sequence $\{x_n\}$ is bounded. For this consider

$$\begin{aligned} d(x_0, x_{2(n+1)}) &\leqslant d(x_0, T_1 x_0) + d(T_1 x_0, T_2 x_{2n-1}) + d(T_1 x_{2n}, T_2 x_{2n-1}) \\ &\qquad + d(x_{2n+1}, x_{2(n+1)}) \\ &\leqslant 3d(x_0, T_1 x_0) + \alpha(d(x_0, x_{2n-1}))\,\{d(x_0, x_1)\, d(x_{2n-1}, x_{2n})\}^{1/2} \\ &\leqslant \{3 + \alpha(d(x_0, x_{2n-1}))\}\, d(x_0, x_1). \end{aligned}$$

Hence for a given $d_0 > 0$ with $d(x_0, x_{2n-1}) \geqslant d_0$, we get

$$d(x_0, x_{2(n+1)}) \leqslant \{3 + \alpha(d_0)\}\, d(x_0, x_1).$$

Similarly we can show that

$$d(x_0, x_{2n+1}) \leqslant \{2 + \alpha(d_0^*)\}\, d(x_0, x_1), \text{ where for a given } d_0^* > 0,$$
$$d(x_0, x_{2n+1}) \geqslant d_0^* .$$

Hence $\{x_n\}$ is bounded. By routine calculation, it follows that for each n,

$$d(x_n, x_{n+1}) \leqslant \{\beta(d(x_{n-1}, x_n))\, \beta(d(x_{n-2}, x_{n-1})) \ldots \beta(d(x_0, x_1))\}\, d(x_0, x_1).$$

where $\beta = \alpha^2$.

Let $\epsilon > 0$. If $d(x_i, x_{i+1}) \geqslant \epsilon$ for $i = 0, 1, 2, \ldots$, then $\beta(d(x_i, x_{i+1})) \leqslant \beta(\epsilon)$ for $i = 0, 1, 2, \ldots$ and also $0 \leqslant \beta(\epsilon) < 1$. Therefore

$$d(x_n, x_{n+1}) \leqslant (\beta(\epsilon))^n\, d(x_0, x_1).$$

This proves that $\{x_n\}$ is a Cauchy sequence. As X is complete, $\lim_{n\to\infty} x_n = \xi \in X$. Using continuity of T_1 and T_2, we find that ξ is a common fixed point of T_1 and T_2. Unicity of ξ is obvious. This ends the proof.

Putting $A = X$ in Theorem 1 we obtain the following Corollary. If X is a complete metric space and if

$$d(T_1x, T_2y) \leqslant \alpha(d(x, y)) \{d(x, T_1x)\, d(y, T_2y)\}^{1/2}$$

for all x, y in X, then T_1 and T_2 have a unique common fixed point.

Theorem 2 — Let T_1 and T_2 be two mappings of a convex, complete metric space (X, d) into itself such that, for $p, q \in S(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ and $0 \leqslant \alpha < 1$, where α and ϵ do not depend on $x \in X$, we have

$$d(T_1p, T_2q) \leqslant \alpha\{d(p, T_1p)\, d(q, T_2q)\}^{1/2}.$$

Then $d(T_1p, T_2q) \leqslant \beta d(p, q)$ for every $p, q \in X$ and some β, $0 \leqslant \beta < 1$.

PROOF : Let $a, b \in X$. Then by a theorem of Menger (see Blumenthal 1953), the convex complete metric space X contains together with a, b, also a metric segment whose extremities are a, b. Using this, we find that if $p, q \in X$, there exist points $p = x_0, x_1, \ldots, x_{n-1} = q$ such that

$$d(p, q) = \sum_{i=1}^{n-1} d(x_{i-1}, x_i) \text{ and } d(x_{i-1}, x_i) < \epsilon.$$

Let us define $\{x_n\}$ by $x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$, ..., $x_{n-1} = T_1x_{n-2}$, $x_n = T_2x_{n-1}$.

Then for an even n,

$$\begin{aligned} d(T_1p, T_2q) &= d(T_1x_0, T_2x_{n-1}) \\ &\leqslant d(T_1x_0, T_2x_1) + d(T_2x_1, T_1x_2) + d(T_1x_2, T_2x_3) \\ &\quad + \ldots + d(T_1x_{n-2}, T_2x_{n-1}) \\ &\leqslant \alpha^2 d(x_0, x_1) + \alpha^2 d(x_1, x_2) + \alpha^2 d(x_2, x_3) \\ &\quad + \ldots + \alpha^2 d(x_{n-2}, x_{n-1}) \\ &= \alpha^2\{d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-2}, x_{n-1})\} \\ &= \alpha^2 d(p, q) \\ &= \beta d(p, q), \quad 0 \leqslant \beta = \alpha^2 < 1. \end{aligned}$$

This completes the proof.

Theorem 2 yields the following:

Corollary — Let T be a self mapping of a convex complete metric space (X, d) such that

$$d(Tp, Tq) \leqslant \alpha\{d(p, Tp)\, d(q, Tq)\}^{1/2}$$

where $p, q \in S(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$, $0 \leqslant \alpha < 1$ and α, ϵ do not depend on $x \in X$. Then T is a contraction mapping.

Theorem 3 — Let X be a Hausdorff space and let T_1 and T_2 be two continuous mappings of X into itself. Let F be a continuous symmetric mapping of $X \times X$ into $[0, \infty)$ such that $F(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$ and

$$F(T_1^p x, T_2^q y) < \{F(x, T_1^p x) F(y, T_2^q y)\}^{1/2}$$

for $x, y \in X$ with $x \neq y$, where $p > 0$, $q > 0$ are integers. If for some $x_0 \in X$, the sequence $\{x_n\}$ where $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$, $n = 0, 1, 2, \ldots$, has a convergent subsequence, then T_1 and T_2 have a unique common fixed point.

PROOF : Similar to Theorem 3 of Ray (1976).

Theorem 4 — Let T be a densifying mapping of a complete metric space (X, d) into itself such that for each pair of distinct $x, y \in X$

$$F(Tx, Ty) < \{F(x, Tx) F(y, Ty)\}^{1/2}$$

where $F : X \times X \to [0, \infty)$ is continuous. If for some $x_0 \in X$, the sequence $\{T^n x_0\}$ is bounded, then T has a unique fixed point.

PROOF : Using standard arguments about densifying mappings (see Furi and Vignoli 1969), we can show that the closure of the set $A = \bigcup_{n=0}^{\infty} \{T^n x_0\}$ is compact and $T : \bar{A} \to \bar{A}$. Then the assertion follows from Theorem 3 with $T_1 = T_2 = T$ and $p = q = 1$.

§3. This last section contains some fixed point theorems in Banach space.

Definition — A mapping T from a Banach space X into itself is said to be 'asymptotically regular' at $x_0 \in X$ if $\| T^n x_0 - T^{n+1} x_0 \| \to 0$ as $n \to \infty$, whenever $T^n x_0$ is defined for all $n \geqslant 1$.

Theorem 5 — Let X be a uniformly convex Banach space, K a subset of X, T a mapping of K into itself such that for all x, y in K

$$\| Tx - Ty \| \leqslant \{\| x - Ty \| \, \| y - Tx \|\}^{1/2}$$

and the set $F(T)$ of fixed points of T is non-empty. If there exists an $x_0 \in K$ and a λ in $(0, 1)$ such that $S^n x_0$ is defined and lies in K for each $n \geqslant 1$, where $S = \lambda I + (1 - \lambda) T$, then S is asymptotically regular at x_0.

PROOF : Let $p \in F(T)$ and $x_0 \in K$ such that $x_n = S^n x_0 \in K$. Obviously $F(T) = F(S)$. Then for all $x_n \in K$, we have

$$\begin{aligned} x_{n+1} - p = S(x_n) - p &= \lambda x_n + (1 - \lambda) T x_n - p \\ &= \lambda(x_n - p) + (1 - \lambda)(Tx_n - Tp). \end{aligned}$$

Hence

$$\| x_{n+1} - p \| \leqslant \lambda \| x_n - p \| + (1 - \lambda) \| Tx_n - Tp \|$$
$$\leqslant \lambda \| x_n - p \| + (1 - \lambda) \| x_n - p \|$$
$$= \| x_n - p \|.$$

Therefore

$$\| x_n - p \| \to p_0 \text{ for some } p_0 \geqslant 0. \quad \text{If } p_0 = 0, \text{ then } x_n \to p$$

and this implies that

$$\| x_n - x_{n+1} \| = \| S^n x_0 - S^{n+1} x_0 \| \to 0, \text{ as required.}$$

For $p_0 > 0$,

$$\| S(x_n) - p \| = \| x_{n+1} - p \| \to p_0 \text{ when } \| x_n - p \| \to p_0.$$

Then it follows from uniform convexity of X that

$$\| (x_n - p) - (S(x_n) - p) \| \to 0 \text{ as } n \to \infty.$$

That is to say

$$\| x_n - S(x_n) \| = \| S^n(x_0) - S^{n+1}(x_0) \| \to 0 \text{ as } n \to \infty.$$

This completes the proof.

Theorem 6 — Let $T : K \to K$ be a densifying mapping defined on a closed, bounded, convex subset of a strictly convex Banach space X and satisfying the following condition

$$\| Tx - Ty \| \leqslant \{ \| x - Ty \| \| y - Tx \| \}^{1/2}$$

for all x, y in K. Then for $x_0 \in K$, the sequence of iterates $\{T_\lambda^n x_0\}$, where $T_\lambda : K \to K$ is defined by $T_\lambda = \lambda I + (1 - \lambda) T$, $0 < \lambda < 1$, converges to a fixed point of T.

PROOF : The following known results will be required in the proof.

Theorem A (Furi and Vignoli 1970) — Let $T : K \to K$ be a densifying mapping defined on a closed, bounded and convex subset of a Banach space X. Then T has at least one fixed point.

Theorem B (Diaz and Metcalf 1969) — Let T be a continuous mapping of a non-empty metric space X into itself. Suppose

(i) the set $F(T)$ of fixed points of T is non-empty,

(ii) for each $x \in X$, with $x \notin F(T)$ and each $p \in F(T)$,

$d(Tx, p) < d(x, p)$.

Let $x_0 \in X$. Then either $\{T^n x_0\}$ has no convergent subsequence or $\{T^n x_0\}$ converges to a fixed point of T.

For the proof of our theorem, first observe that T_λ is a densifying mapping. Moreover, $F(T)$ and $F(T_\lambda)$ coincide and by Theorem A, $F(T)$ and hence, $F(T_\lambda)$ is non-empty. As indicated in the proof of Theorem 4, we can show that the closure of the set

$$A = \bigcup_{n=0}^{\infty} T_\lambda^n x_0$$

is compact and $T_\lambda : \bar{A} \to \bar{A}$. Hence the sequence of iterates has a convergent subsequence. Let $p \in F(T) = F(T_\lambda)$ and $x \in K - F(T)$. Then by the condition of the theorem $\| Tx - p \| \leqslant \| x - p \|$. Now for $x \neq p$,

$$\| T_\lambda x - p \| = \| T_\lambda x - T_\lambda p \| = \| \lambda(Tx - p) + (1 - \lambda)(x - p) \|$$

$$= \| x - p \| \left\{ \frac{\lambda(Tx - p)}{\| x - p \|} + (1 - \lambda) \frac{(x - p)}{\| x - p \|} \right\}.$$

Since X is strictly convex and $\dfrac{(Tx - p)}{\| x - p \|}$ and $\dfrac{(x - p)}{\| x - p \|}$ have unit norm, we get

$$\left\| \lambda \frac{(Tx - p)}{\| x - p \|} + (1 - \lambda) \frac{(x - p)}{\| x - p \|} \right\| < 1.$$

Thus $\| T_\lambda x - p \| < \| x - p \|, x \neq p$.

Now the result follows from Theorem B.

Theorem 7 — Let K be a closed subset of strictly convex Banach space X and let $T : K \to K$ be a continuous mapping satisfying

$$\| Tx - Ty \| \leqslant \{ \| x - Ty \| \, \| y - Tx \| \}^{1/2}$$

for all $x, y \in K$. If $T(K)$ is contained in a compact subset K_1 of K, then for every $x_0 \in K$, the sequence of iterates $\{T_\lambda^n x_0\}$ where $T_\lambda : K \to K$ is defined by

$$T_\lambda = \lambda I + (1 - \lambda) T, \quad 0 < \lambda < 1,$$

converges to a fixed point of T.

Proof : Here $F(T) = F(T_\lambda)$ and $F(T) \neq \phi$ by Schauder's fixed point theorem. Hence $F(T_\lambda) \neq \phi$. Since $T(K) \subseteq K_1$, we find that $\alpha(T(K)) = 0$ which implies that T is completely continuous hence densifying. Applying Theorem 6, we get the result.

Definition (Brodski and Milman 1948) — A bounded convex subset K of a Banach space is said to have 'normal structure' if for each convex subset H of K with more than one point, there is a point $x \in H$ such that

$$\text{Sup}\,\{\| x - y \| : y \in H\} < \delta(H)$$

where $\delta(H)$ is the diameter of H.

By $O(x)$ we denote the orbit $\{x, Tx, T^2x, \ldots\}$ of $T : K \to K$. Finally Co (K) and $\overline{\text{Co}\,(K)}$ will denote the convex hull and closed covex hull of K. In the following theorem we follow the same line of proofs as given by Kirk (1970).

Theorem 8 — Let K be a non-empty, bounded, closed and convex subset of a reflexive Banach space X and let K have normal structure. If T is a mapping of K into itself such that

$$\| Tx - Ty \| \leqslant \{\| x - Tx \| \, \| y - Ty \|\}^{1/2}$$

for all $x, y \in X$, then T has a unique fixed point.

The proof of the following lemma is simple.

Lemma — Let K be a subset of a Banach space and let T be a mapping of K into itself, such that for $x, y \in K$, T satisfies

$$\| Tx - Ty \| \leqslant \{\| x - Tx \| \, \| y - Ty \|\}^{1/2}.$$

Then for $x \in K$, and positive integers m and n,

$$\| T^n x - T^{n+1} x \| \leqslant \| T^{n-1} x - T^n x \|$$

and

$$\| T^m x - T^n x \| \leqslant \| x - Tx \|.$$

PROOF OF THEOREM 8 : Reflexivity of X implies that every descending chain of non-empty closed convex subsets of X has non-empty intersection (Kirk 1965). Hence by Zorn's lemma, there is a minimal subset K_1 of K with respect to being closed, covex and invariant under T. If $\delta(K_1) = 0$, we are done. So let $\delta(K_1) > 0$. Since K has a normal structure, there is a point $y \in K$, such that

$$\text{Sup}\,\{\| x - y \| : x \in K_1\} \leqslant r < \delta(K_1).$$

Then $\| Ty - y \| \leqslant r$ and our lemma implies that $\delta(0(y)) \leqslant r$. Let

$$N = \{x \in K_1 : \| x - Tx \| \leqslant r\} \text{ and } P = \text{Co}\,(T(N)).$$

Then P is closed, convex and non-empty. We wish to show that $T : P \to P$. Let $y \in P$. Then there are three cases to consider.

Case 1 — $y = Tp$ for some $p \in N$. By lemma,

$$\| Ty - y \| = \| T^2 p - Tp \| \leqslant \| p - Tp \| \leqslant r.$$

Case 2 — Let $y = \sum_{i=1}^{n} \lambda_i T p_i$, $p_i \in N$, $\lambda_i \geqslant 0$, $\sum_{i=1}^{n} \lambda_i = 1$.

Then

$$\| Ty - y \| = \left\| Ty - \sum \lambda_i Tp_i \right\|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \| Ty - Tp_i \|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \{ \| y - Ty \| \| p_i - Tp_i \| \}^{1/2}$$

$$\leqslant \{ r \| y - Ty \| \}^{1/2}.$$

So

$$\| Ty - y \| \leqslant r \text{ giving thereby } Ty \in P.$$

Case 3 — y is the limit of terms of the form

$$\sum_{i=1}^{n} \lambda_i Tp_i \text{ for } p_i \in N, \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1.$$

Then

$$\| Ty - y \| \leqslant \left\| Ty \leqslant \sum_{i=1}^{n} \lambda_i Tp_i \right\| + \left\| \sum_{i=1}^{n} \lambda_i Tp_i - y \right\| \cdot$$

By Case 2, and as $n \to \infty$ we get

$$\| Ty - y \| \leqslant r.$$

Hence $Ty \in P$. But minimality of K_1 gives $P = K_1$.

Now,

$$\delta(K_1) = \delta(P)$$

$$= \delta\overline{(\mathrm{Co}\,(T(N))}$$

$$= \delta(T(N))$$

$$= \mathrm{Sup}\,\{ \| Tx - Ty \| : x, y \in N \}$$

$$\leqslant \mathrm{Sup}\,\{ \| x - Tx \| \| y - Ty \| \}^{1/2} : x, y \in N \}$$

$$\leqslant r < \delta(K_1).$$

Thus, $\delta(K_1) > 0$ leads to a contradiction. Hence $\delta(K_1) = 0$ and T has a fixed point which is obviously unique.

Theorem 9 — Let X be a strictly convex reflexive Banach space and K a bounded, closed and convex subset of X. Let T be a mapping of K into itself such that for all x, y in K

$$\| Tx - Ty \| \leqslant \{\| x - Tx \| \, \| y - Ty \|\}^{1/2}.$$

Then T has a unique fixed point.

PROOF: As in the proof of Theorem 8, we can obtain a subset K_1 of K which is minimal with respect to being closed, convex and invariant under T. Suppose $\delta(K_1) > 0$. Let $x \in K_1$ and assume that $\| x - Tx \| = \delta(K_1)$. Take $y = \frac{1}{2}(x + Tx)$. Then $y \in K_1$ and hence

$$\| x - Ty \| \leqslant \delta(K_1) \quad \text{and} \quad \| Ty - Tx \| \leqslant \delta(K_1).$$

By strict convexity of X, we have $\frac{1}{2} \| Ty - x + Ty - Tx \| < \delta(K_1)$. So

$$\| Ty - y \| < \delta(K_1)$$

and there exists a point $y \in K_1$ with $\| Ty - y \| = r < \delta(K_1)$. The rest of the proof is identical with the proof of Theorem 8.

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