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ON ENTIRE FUNCTIONS OF BOUNDED INDEX DEFINED BY DIRICHLET EXPANSIONS

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Recently, Azpeitia (1977) has extended the notion of bounded index to entire functions represented by Dirichlet series and has obtained a parallel result due to Shah and Sisarcick (1971) for this case. In this note, we extend this idea in more general form and obtain some more results in this direction.

§1. An entire function $f(s)$ which admits a Dirichlet expansion of the form

$$f(s) = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n); \; \lambda_0 \geqslant 0, \; \lambda_{n+1} > \lambda_n \qquad \ldots(1)$$

absolutely convergent everywhere and such that $\liminf_{n=\infty} (\lambda_{n+1} - \lambda_n) > 0$ is said to be of bounded index if and only if there is an integer ν such that for all s

$$\max \{ \mid f^{(j)}(s) \mid / j! \bigm| 0 \leqslant j \leqslant \nu \} \geqslant \sup \{ \mid f^{(j)}(s) \mid / j! \bigm| j = 0, 1, 2, \ldots \} \qquad \ldots(2)$$

where $f^{(0)}(s)$ stands for $f(s)$. The smallest of all the integers with such property is called the index of $f(s)$. Recently, Azpeitia (1977) proved the following.

Theorem A — If $f(s)$ is of bounded index N, then it reduces to an exponential polynomial (i.e., $a_n = 0$ from some n on).

This parallels a result for the case of ordinary Taylor series due to Shah (1968, Theorem 1). In this note, we shall consider a slight variation of (2), namely that for some integer $N \geqslant 0$, some $C > 0$, some $\sigma_0 > 0$ and for all s with Re $\{s\} > \sigma_0$

$$\sum_{j=0}^{N} \mid f^{j}(s) \mid / j! > C \sum_{j=N+1}^{\infty} \mid f^{j}(s) \mid / j!. \qquad \ldots(3)$$

We also consider the variation of (3) by

$$\sum_{j=0}^{N} \frac{\left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f^{j}(\sigma + it) |^{2} dt\right\}}{j!}$$

$$> C \sum_{j=N+1}^{\infty} \frac{\left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f^{j}(\sigma + it) |^{2} dt\right\}}{j!} \qquad \ldots(4)$$

and also

$$\sum_{j=0}^{N} \frac{M(\sigma, f^{(k+j)}(s))}{j!} > C \sum_{j=N+1}^{\infty} \frac{M(\sigma, f^{(k+j)}(s))}{j!} \qquad \ldots(5)$$

where $M(\sigma, f^{(k+j)}(s)) \equiv M(\sigma, f^{k+j}) = \max_{-\infty<t<\infty} | f^{k+j}(s) |$, and $s = \sigma + it$. The smallest of all the integers with the property (3) {and similarly in (4) and (5)} is called the non-uniform index of $f(s)$. It is interesting to note that if $f(s)$ is of bounded index N [cf. (2)], then it satisfies for all s

$$\sum_{j=0}^{N} \frac{| f^{j}(s) |}{j!} \geqslant \frac{| f^{j}(s) |}{j!}; \; j = 0, 1, 2, \ldots \qquad \ldots(6)$$

and conversely, if (6) is satisfied then

$$\max\left\{ | f(s) |, \frac{| f'(s) |}{1!}, \ldots, \frac{| f^{N}(s) |}{N!}\right\} \geqslant \left(\frac{1}{N+1}\right) \frac{| f^{j}(s) |}{j!}; \quad j = 0, 1, \ldots . \qquad \ldots(7)$$

Thus, we can define, the smallest of integer N to be related bounded index of $f(s)$, if the following is satisfied for all s

$$\max\left\{ | f(s) |, \frac{| f'(s) |}{1!}, \ldots, \frac{| f^{N}(s) |}{N!}\right\} \geqslant C\left\{\frac{| f^{j}(s) |}{j!}\right\} \qquad \ldots(8)$$

for $j = 0, 1, 2, \ldots$ and $C > 0$.

§2. We prove the following:

Theorem — If $f(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$, $\lambda_0 \geqslant 0$, $\lambda_n < \lambda_{n+1} \to \infty$ be absolutely convergent such that $\liminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) > 0$, then $f(s)$ reduces to exponential polynomial (i.e., $a_n = 0$ from some n on) whenever there exist two nonnegative integers k and N independent of s for which either (3) or (4) or (5) or (8) is satisfied.

PROOF : Let (3) be true and assume that expansion in (1) is infinite (i.e., infinity of a_n's in (1) are non-zero). Then, if we write $s = \sigma + it$ and define the maximum term in the usual form

$$m(\sigma) = \max \{ \mid a_n \mid \exp(\sigma\lambda_n) \mid n \geqslant 0\} \quad ...(9)$$

it is known (Azpeitia 1961, p. 717) that index function

$$n(\sigma) = \max \{n \mid m(\sigma) = \mid a_n \mid e^{\sigma\lambda_n}\} \quad ...(10)$$

is monotonically divergent to $+\infty$ with σ and that the same is true for the exponent index function $\lambda_{n(\sigma)}$, i.e.,

$$\lim_{\sigma\to\infty} \lambda_{n(\sigma)} = \infty. \quad ...(11)$$

From (11), it follows that

$$\lambda_{n(\sigma)} > b \text{ for } \sigma_n \leqslant \sigma \leqslant c + \sigma_n < \sigma_{n+1};\ c > 0 \quad ...(12)$$

for any arbitrarily large number b and $n \geqslant n_0$. On the other hand, if we define

$$M_j(\sigma) = \max_{-\infty<t<\infty} \mid f^j(\sigma + it) \mid \ ;\ j = 0, 1, 2, ... \quad ...(13)$$

then, it is also known (Sunyer 1961, Theorem A and Theorem 2) that for any sequence of values of σ tending to infinity in the complement S of the union of the exceptional sets of the functions $f^0, f^1, ..., f^{N+1}$ of finite variation, we have

$$\lim_{j\to\infty} \left[\frac{M_j(\sigma)}{M_k(\sigma)}\right] [\lambda_{n(\sigma)}]^{k-j} = 1;\ 0 \leqslant j, k \leqslant N + 1. \quad ...(14)$$

But (14) is equivalent to

$$\frac{M_j(\sigma)}{M_k(\sigma)} \approx \{\lambda_{n(\sigma)}\}^{j-k};\ 0 \leqslant j, k \leqslant N + 1. \quad ...(15)$$

Also, from (3), it follows that

$$C \frac{M_{N+p}(\sigma)}{(N + p)!} \leqslant \sum_{j=0}^{N} \frac{M_j(\sigma)}{j!};\ p \geqslant 1. \quad ...(16)$$

Now, let $E_n = [\sigma_n, c + \sigma_n]$ and write $G = \bigcup_{n=1}^{\infty} E_n$. The variation of σ in E_n is $c(> 0)$ and so on G is infinite. Thus the set $G \cap S$ contains the points $\sigma_n \to \infty$ for some $n > n_0$. Hence, for $n > n_0$ and $\sigma = \sigma_n$, we have from (12), (15) and (16)

$$CM_{N+j}(\sigma) < (N + j)! \sum_{i=0}^{N} \frac{M_{N+j}(\sigma)}{i!} \left\{\frac{M_i(\sigma)}{M_{N+j}(\sigma)}\right\}$$

(*equation continued on p.* 425)

$$\simeq (N+j)! \sum_{i=0}^{N} \frac{M_{N+i}(\sigma)}{i!} \left\{\frac{1}{\lambda_{n(\sigma)}}\right\}^{N+j-i}$$

$$< (N+j)! \sum_{i=0}^{N} \frac{M_{N+i}(\sigma)}{i!} \frac{1}{b_1^{N+j-i}} \qquad ...(17)$$

where $b_1 < b$, $1 \leqslant j \leqslant N$.

Thus, the last expression gives us

$$\frac{N!\, Cb_1^j}{(N+j)!} < 1 + \frac{N}{b_1} + \frac{N(N-1)}{b_1^2} + ... + \frac{N!}{b_1^N}. \qquad ...(18)$$

Since in (18), $b_1 < b$ is arbitrarily large, $C > 0$ and N is finite, it follows that right-hand side is near 1 whereas the left-hand side is near infinity. This is a contradiction. Hence, we must have expansion in (1) to be finite. This completes the proof. We also note that, if instead of (3), we let (5) to be true then (16) immediately follows and therefore above proof implies that $f(s)$ is an exponential polynomial. In the case when (8) holds true, the proof is similar and so we omit it. Lastly, we assume (4) is true. If we write

$$I_2(\sigma, f^j) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f^j(\sigma + it) |^2 \, dt$$

then (4) takes the form

$$\sum_{j=0}^{N} \frac{I_2(\sigma, f^j)}{j!} > C \sum_{j=N+1}^{\infty} \frac{I_2(\sigma, f^j)}{j!}. \qquad ...(19)$$

Using a result of Gupta (1964) (see also Rahman 1966), we have

$$I_2(\sigma, f^1) \geqslant I_2(\sigma, f) \left\{\frac{\log I_2(\sigma, f)}{2\sigma}\right\}^2 \qquad ...(20)$$

and inductively

$$I_2(\sigma, f^j) \geqslant I_2(\sigma, f^{j-1}) \left\{\frac{\log I_2(\sigma, f^{j-1})}{2\sigma}\right\} \qquad ...(21)$$

for all $j \geqslant 1$.

Now, suppose that $f(s)$ defined by (1) possess an infinite expansion, i.e., infinity of a_n's are non-zero. Then, clearly (11) is true along with

$$\lim_{\sigma\to\infty} \lambda_{n(\sigma,f^j)} = \infty;\ j = 0, 1, 2, \ldots, f^0 \equiv f \quad \ldots(22)$$

where $\lambda_{n(\sigma,f^j)} = \max \{\lambda_n \mid m(\sigma, f^j) = |\ a_n\ |\ \lambda_n^j \exp (\sigma\lambda_n)\}$.

Also, by a result of Yu (1951) we have

$$\log m(\sigma, f^j) = \log m(\sigma_0, f^j) + \int_{\sigma_0}^{\sigma} \lambda_{n(\sigma,f^j)}\, d\sigma$$

$$\geqslant \lambda_{n(\sigma/k,f^j)}\left(1 - \frac{1}{k}\right)\sigma \quad \ldots(23)$$

for $1 < k < \infty$. From (22) and (23) it follows that

$$\log m(\sigma, f^j) \geqslant \left(1 - \frac{1}{k}\right)\sigma b \quad \ldots(24)$$

for $\sigma_n \leqslant \sigma \leqslant c + \sigma_n < \sigma_{n+1}$ and for $j = 1, 2, \ldots$ with b arbitrarily large. Also, obviously, we have

$$I_2(\sigma, f^j) = \sum_{n=0}^{\infty} |\ a_n\ |^2 \lambda_n^j \exp (2\sigma\lambda_n) \geqslant (m(\sigma, f^j))^2. \quad \ldots(25)$$

From (24) and (25), we get

$$\log I_2(\sigma, f^j) \geqslant 2 \log m(\sigma, f^j) \geqslant 2\sigma b \left(1 - \frac{1}{k}\right). \quad \ldots(26)$$

Thus, (21) takes the form

$$I_2(\sigma, f^{j+1}) \geqslant I_2(\sigma, f^j)\, b \left(1 - \frac{1}{k}\right);\ j = 0, 1, 2, \ldots . \quad \ldots(27)$$

Now, take $k = 2$ and use (27) in (19) to get

$$S = \sum_{j=0}^{N} \frac{I_2(\sigma, f^j)}{j!} > C\left[\left(\sum_{j=N+1}^{2N} + \sum_{2N+1}^{3N} + \sum_{3N+1}^{4N} + \ldots\right)\frac{I_2(\sigma, f^j)}{j!}\right]$$

$$> C\left[\frac{I_2(\sigma, f^{N+1})}{(N+1)!} + \frac{I_2(\sigma, f^{N+2})}{(N+2)!} + \ldots + \frac{I_2(\sigma, f^{2N})}{(2N)!}\right]$$

$$> C\left(\frac{b}{2}\right)^{N+1}\left[\frac{I_2(\sigma, f)}{(N+1)!} + \frac{I_2(\sigma, f^1)}{(N+2)!} + \ldots + \frac{I_2(\sigma, f^N)}{(2N)!}\right]$$

$$\geqslant \frac{C(b/2)^{N+1}}{(2N)!}\left[I_2(\sigma, f) + \frac{I_2(\sigma, f^1)}{1!} + \ldots + \frac{I_2(\sigma, f^N)}{N!}\right]$$

$$= \left[\frac{C(b/2)^{N+1}}{(2N)!}\right] S. \quad \ldots(28)$$

This, last inequality gives us,

$$1 \geqslant \left[\frac{C}{2^{N+1}}\right] \frac{b^{N+1}}{(2N)!}.$$

Since b is arbitrarily large and $N \geqslant 0$, it follows that right-hand side goes to infinity as $b \to \infty$. This is a contradiction. Hence $f(s)$ must be an exponential polynomial.

A parallel result for the case of ordinary Taylor series is due to Shah and Sisarcick (1971) and Gross (1970).

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