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A NOTE ON BOUNDED INDEX AND BOUNDED VALUE DISTRIBUTION

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The class of entire functions of bounded value distribution is not closed under multiplication not even under multiplication by exp (z). A necessary and sufficient condition for an entire function of finite order to be of bounded index is given.

An entire function f is said to be of bounded value distribution (b.v.d.) if for every $r > 0$ there exists a fixed integer $C(r) > 0$ such that the equation $f(x) = w$ has never more than $C(r)$ roots in any disk of radius r and for any $w \in \mathbb{C}$ (see Hayman 1967, problem 2.28).

An entire function f is said to be of bounded index (b.i.) if there exists an integer $N \geqslant 0$ such that

$$\max_{0 \leqslant j \leqslant N} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \geqslant \frac{|f^{(n)}(z)|}{n!}$$

for all $z \in \mathbb{C}$ and all $n = 0, 1, \ldots$. The least such integer N is called the index of f. The concept of b.i. was first introduced by Lepson (1968).

Hayman (1973) proved that an entire function f is of b.v.d. if and only if f' is of b.i. Since functions of b.i. are not closed under addition (Pugh 1969) we easily see that the same applies to functions of b.v.d. Although functions of b.i. are closed under multiplication (Fricke 1973), the result cannot be extended to functions of b.v.d. In fact we will show that functions of b.v.d. are not closed under multiplication by e^z.

Theorem 1 — There exists an entire function $f(z)$ of b.v.d. such that $h(z) = e^z f(z)$ is not of b.v.d.

PROOF : Suppose the class of functions of b.v.d. is closed under multiplication by e^z. Shah (1971) gave an example of a function g of u.b.i. such that $g(z) + c$ is of b.i. for any non-zero complex number c. In particular $g(z) + 1$ is of b.i. and thus $l(z) = e^z(g(z) + 1)$ is also of b.i. Let $G(z)$ be an antiderivative of $e^z g(z)$, then $L(z) = G(z) + e^z$ is an antiderivative of $l(z)$ and therefore of b.v.d. Since we assumed closure under multiplication by e^z we have $e^{-z}L(z) = e^{-z}G(z) + 1$ is of b.v.d. Therefore $e^{-z}G(z)$ and also $G(z) = e^z(e^{-z}G(z))$ have to be of b.v.d. Thus

$G'(z) = e^z g(z)$ is of b.i. which contradicts the fact that $g(z) = e^{-z}G'(z)$ is of u.b.i.

q.e.d.

In an earlier paper the author (Fricke 1973) proved:

Theorem A — Let f be an entire function of exponential type. Then f is of b.i. if and only if for each $d > 0$ there exists $M = M(d) > 0$ such that $|f'(z)| \leqslant M |f(z)|$ for all z with $|z - a_n| \geqslant d$ for all n, where the a_n's are the zeros of f.

The bound M depends on the closeness to the zeros of f. If we now examine the behaviour of the logarithmic derivative near a zero a_t of order m we see that

$$\frac{f'(z)}{f(z)} - \frac{m}{z - a_t}$$

is bounded in a neighbourhood of a_t. We will use this simple observation to improve the above theorem. For an entire function f let

$$R(z) = R_f(z) = \sup \{1\} \cup \left\{ \frac{1}{|z - a_n|}, n = 1, 2, \ldots \right\}$$

where the a_n's are the zeros of f. We then show:

Theorem 2 — An entire function f of finite order is of b.i. if and only if there exists a costants $M > 0$ such that $|f'(z)| \leqslant MR(z) |f(z)|$ for all z.

For the proof of Theorem 2 we need to establish the following lemma.

Lemma — If f is an entire function of finite order such that for some $M > 0$, $|f'(z)| \leqslant MR(z) |f(z)|$ for all z then there exists an integer N such that any closed disc of radius 1 contains at most N zeros of f.

Proof: Let f be of order ρ and let the a_n's denote the zeros of f. Let $n(r, w)$ denote the number of zeros of f in $\{z \,|\, |z - w| \leqslant r\}$. It is well known (Boas 1954, Theorem 25.12) that for any $\epsilon > 0$,

$$n(r) = n(r, 0) \leqslant r^{\rho+\epsilon} \qquad \ldots(1)$$

provided r is sufficiently large.

Now, for a given $w \in \mathbb{C}$, let $n(1, w) = N$ and choose $\lambda > \rho$, then let k be largest integer t such that

$$n(t, w) \geqslant N - 2 \cdot 4^{\lambda+1} + 2t^{\lambda+1}. \qquad \ldots(2)$$

Because of (1) such an integer k exists and obviously $k \geqslant 4$. Then

$$n(k + 4, w) - n(k, w) \leqslant 2(k + 4)^{\lambda+1} - 2k^{\lambda+1} \leqslant Lk^{\lambda},$$

where L is a constant independent of k and w.

We now subdivide the annulus $\{z \mid k < |z-w| \leqslant k+4\}$ into k equal sections $\left\{A_j\right\}_{j=1}^{k}$ where

$$A_j = \left\{z \,\middle|\, k < |z-w| \leqslant k+4 \text{ and } (j-1)\frac{2\pi}{k} \leqslant \arg(z-w) < \frac{2\pi}{k}\right\}.$$

To simplify notation we define $A_k = A_0$, $A_{k+1} = A_1$, $A_{k+2} = A_2$ and $A_{k+3} = A_3$. Then let B_j denote the number of zeros of f in A_j for $j = 0, 1, \ldots, k+3$. For each j define an arc R_j in $A_j \cup A_{j+1}$ by $R_j = \left\{z \,\middle|\, |z-w| = t_j \text{ and } (j-1)\frac{2\pi}{k} \leqslant \arg z < (j+1)\frac{2\pi}{k}\right\}$, where $k+1 \leqslant t_j \leqslant k+3$. Then for

$$a_n \notin A_{j-1} \cup A_j \cup A_{j+1} \cup A_{j+2},\ |z-a_n| \geqslant 1$$

for all $z \in R_j$. Hence it is possible to choose t_j such that for all

$$z \in R_j,\ |z-a_n| \geqslant (1+B_{j-1}+B_j+B_{j+1}+B_{j+2})^{-1}$$

for all n. Since R_j and R_{j+1} run completely through A_{j+1} we can connect R_j and R_{j+1} by a line $C_j \subset A_{j+1}$ with length not exceeding 2 such that for all $z \in C_j$, $|z-a_n| \geqslant (1+B_{j+1})^{-1}$ for all n.

Using the C_j's and the appropriate portions of the R_j's we can define a simple closed path $\gamma(t)$ about w lying in $\{z \mid k+1 \leqslant |z-w| \leqslant k+3\}$ with the following properties:

(i) $$|\gamma(t) \cap A_j| \leqslant |R_{j-1}| + |C_{j-1}| + |R_j| \leqslant \frac{2\pi}{k}(k+3) + 2 + \frac{2\pi(k+3)}{k} < 30,$$ where $|\cdot|$ denotes path length.

(ii) $|z-a_n| \geqslant (1+B_{j-1}+B_j+B_{j+1}+B_{j+2})^{-1}$ for all n and for all $z \in \gamma \cap A_j$.

Hence, for $z \in \gamma \cap A_j$, $\left|\frac{f'(z)}{f(z)}\right| \leqslant MR(z) \leqslant M(1+B_{j-1}+\ldots+B_{j+2})$. Thus we can obtain the following estimate on the number of zeros of f inside $\gamma(t)$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz \leqslant \frac{1}{2\pi}\int_\gamma \left|\frac{f'(z)}{f(z)}\right| |dz|$$
$$\leqslant \sum_{j=1}^{k} 30M(1+B_{j-1}+B_j+B_{j+1}+B_{j+2})$$

(*equation continued on p.* 431)

$$= \sum_{i=1}^{k} 30M(1 + 4B_i)$$

$$\leqslant 30M(k + 4n(k+4, w) - 4n(k, w))$$

$$\leqslant 30M(k + 4Lk^\lambda).$$

Now, $n(k, w) \leqslant n(\gamma, w) \leqslant 30M(k + 4Lk^\lambda)$ and by (2)

$$N - 2.4^{\lambda+1} + 2k^{\lambda+1} \leqslant 30M(k + 4Lk^\lambda) \text{ and}$$

$$N \leqslant 30Mk + 120MLk^\lambda + 8^{\lambda+1} - 2k^{\lambda+1}.$$

For $x \geqslant 0$ let $g(x) = 30M + 120MLx^\lambda + 8^{\lambda+1} - 2x^{\lambda+1}$, then $\lim_{x\to\infty} g(x) = -\infty$ implies g is bounded above. Hence, N is bounded independently of w. q.e.d.

Proof of Theorem 2 — "$\Leftarrow$" Let f be of finite order and let $M > 0$ be a constant such that $|f'(z)| \leqslant MR(z)\,|f(z)|$ for all z. Then by the previous lemma any disk of radius one contains at most N zeros of f for some integer N.

Consider all line segments s satisfying

(i) $|z| \leqslant r + 2$ for all $z \in s$

(ii) $|z - a_n| \geqslant \dfrac{1}{2N+2}$ for all n and all $z \in s$

(iii) length of $s \geqslant 1$.

If A is the union of all those line segments, then A is connected and

$$\max_{|z| \leqslant r} \{|f(z)|\} \leqslant \max_{z \in A} \{|f(z)|\}.$$

Also for any line segments there exists a path of length not exceeding $3(r+2)$ connecting s and the unit disk, where the path consists of pieces of line segments of above type. Let $\gamma(t)$ be such a path, then

$$\left|\int_\gamma \frac{f'(z)}{f(z)}\,dz\right| \leqslant M \max_{z\in\gamma} \{R(z)\}\,|\gamma| \leqslant M(2N+2)\,3(r+2).$$

Thus

$$\max_{|z|=r} \{|f(z)|\} \leqslant \exp\{M(2N+2)(3r+6)\} \max_{|z|\leqslant 1} \{|f(z)|\}.$$

Hence, f is of exponential type not exceeding $3M(2N+2)$. If $|z - a_n| \geqslant d$ for all n then $R(z) \leqslant \dfrac{1}{d}$ and $|f'(z)| \leqslant M/d\,|f(z)|$ and thus by Theorem A, f is of b.i.

"$\Rightarrow$" Let f be of b.i. then there exists an integer K such that any disk of radius 1 contains at most K zeros of f (see Hayman 1973). Also, for $d = 1/(2K + 2)$ we have by Theorem A, there exists a constant M such that $|f'(z)| \leqslant M |f(z)|$ for z with $|z - a_n| \geqslant 1/(2K + 2)$.

For each $w \in ¢$ there exists r_w with $d \leqslant r_w \leqslant 1$ such that $|z - a_n| \geqslant d$ for all n and for all z with $|z - w| = r_w$ (Choose $r_w = id$ for some $1 \leqslant i \leqslant 2K + 2$). Then, $\left|\frac{f'(z)}{f(z)}\right| \leqslant M$ for all z with $|z - w| = r_w$. Let $b_1, b_2, \ldots, b_l$ be the zeros of f in $\{z \mid |z - w| < r_w\}$. Obviously $l \leqslant K$ and $f(z) = \prod_{j=1}^{l} ((1 - z/b_j))\, g(z)$, where g is entire and $g(z) \neq 0$ in $\{z \mid |z - w| \leqslant r_w\}$. Hence,

$$\left|\frac{f'(w)}{f(w)}\right| = \left|\frac{g'(w)}{g(w)} + \sum_{j=1}^{l} \frac{1}{w - b_j}\right| \leqslant \max_{|z-w|=r_w} \left|\frac{g'(z)}{g(z)}\right| + lR(w)$$

$$= \max_{|z-w|=r_w} \left\{\left|\frac{f'(z)}{f(z)} - \sum_{j=1}^{\infty} \frac{1}{z - b_j}\right|\right\} + lR(w)$$

$$\leqslant \max_{|z-w|=r_w} \left\{\left|\frac{f'(z)}{f(z)}\right| + \sum_{j=1}^{l} \frac{1}{|z - b_j|}\right\} + KR(w)$$

$$\leqslant M + \frac{K}{d} + KR(w)$$

$$\leqslant \left(M + \frac{K}{d} + K\right) R(w) = QR(w), \text{ where } Q \text{ is independent of } w.$$

q.e.d.

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