Indian J. pure appl. Math., 11(4) : 433–437, April 1980

THE GEOMETRICAL RELATIONSHIP BETWEEN THE QUADRUPLE $(P_{n+1}(C), Q_n(C), P_{n+1}(R), V_{n+2,2}(R))$

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(*Received* 6 *August* 1979)

We study the relations between the $(n+1)$-complex projective space $P_{n+1}(C)$, the $(n+1)$-real projective space $P_{n+1}(R)$, the n-complex quadric $Q_n(C)$ in $P_{n+1}(C)$ and the real Stiefel manifold $V_{n+2,2}(R)$ and prove a few theorems.

1. INTRODUCTION

An $(n+1)$-complex projective space $P_{n+1}(C)$ is defined as the space of complex lines through the origin in C^{n+2}. Let $(z_0, z_1, \ldots, z_{n+1})$ be homogeneous complex coordinates on $P_{n+1}(C)$ and left f be a differentiable function on $P_{n+1}(C)$ defined by

$$f(z_0, z_1, \ldots, z_{n+1}) = \Big(\sum_{j=0}^{n+1} z_j\Big)\Big(\sum_{j=0}^{n+1} \bar{z}_j\Big)\Big/\Big(\sum_{j=0}^{n+1} \mid z_j^2 \mid\Big)^2.$$

An $(n+1)$-real projective space $P_{n+1}(R)$ is defined as the space of real lines through the origin in R^{n+2}, the n-complex quadric $Q_n(C)$ in $P_{n+1}(C)$ is given by the equation

$$\sum_{j=0}^{n+1} z_j^2 = 0$$

and the real Stiefel manifold $V_{n+2,2}$ is defined as the set of all 2-frames in R^{n+2} (an ordered set of 2-linearly independent vectors of R^{n+2}) endowed with induced differentiable structure (see Brickell and Clark 1970)

In this paper we study the geometrical relationship between the quadruple $(P_{n+1}(C), Q_n(C), P_{n+1}(R), V_{n+2,2}(R))$. Specifically we prove the following:

Theorem 1 — X is diffeomorphic to $V_{n+2,2}(R) \times (0, 1)$, where X denotes the connected non-compact manifold $P_{n+1}(C) - Q_n(C) - P_{n+1}(R)$.

On the other hand if $B = X/SO(n+2)$ we can prove the following series of lemmas:

Lemma 1 — B has a differentiable structure and diffeomorphic to (0, 1).

Lemma 2 — If $a, b \in B$ provided that $a \neq b$ then $f(P) \neq f(q)$ for all $p \in a$ and $q \in b$.

Finally we prove the following two theorems:

Theorem 2 — (i) If $f(p) = 0$, for all $p \in P_{n+1}(C)$, then the level surfaces (see Definition 2.3) of f are homeomorphic to $Q_n(C)$.

(ii) If $f(p) = 1$, for all $p \in P_{n+1}(C)$, then the level surfaces of f are homeomorphic to $P_{n+1}(R)$.

(iii) If $0 < f(p) < 1$, for all $p \in P_{n+1}(C)$, then the level surfaces of f are homeomorphic to $V_{n+2,2}(R)$.

Theorem 3 — $P_{n+1}(C) - P_{n+1}(R)$ is diffeomorphic to the normal bundle of $Q_n(C)$ in $P_{n+1}(C)$.

2. Preliminaries (Bott 1960)

Let M be a compact differentiable n-manifold and

$$g : M \to R$$

a differentiable real-valued function on M. Since dim. $R = 1$ and dim $M \geqslant 1$; it follows that the Rank $r_p g$ of g at $P \in M$, i.e., the rank of the linear transformation $dg : M_p \to R$ induced by the tangent spaces, is either 0 or 1.

Definition 2.1 — If $r_p g = 0$, then p is called a critical point of g.

Definition 2.2 — If $r_p g = 1$, then p is called a regular point of g.

Definition 2.3 — For each $a \in R$, the set $g^{-1}(a)$ is called the a-level surface of g in M.

Definition 2.4 — The set $[p : p \in M; g(p) \leqslant a$, for each $a \in R]$ will be denoted by $g^a M$, or just by M^a if the function g is understood, and is called a half-space for g on M.

Let f and X be as stated above. Suppose that $a, b \in f(P_{n+1}(C))$ as $f^{-1}(a)$, $f^{-1}(b) \subset X$. Then it is well known that both $f^{-1}(a)$ or $f^{-1}(b)$ are submanifolds of $P_{n+1}(C)$ with codimension one. Let p be either a point of $f^{-1}(a)$ or $f^{-1}(b)$. Thus p is *a* regular point of f (see Definition 2.2). Since X is an open submanifold of $P_{n+1}(C)$, we can choose a neighbourhood N (say) of p such that NX. Given a chart $\phi : N \to C^{n+2}$ on N and suppose, without any loss of generality, that $\phi(p) = o$ (o denotes the origin in C^{n+2}). Write $F = f\phi$ which is so clear that F has rank one everywhere. Hence $\partial F(o)/\partial x_j \neq 0$ and $\partial F(o)/\partial y_j \neq 0$, where $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$ are the underlying real coordinate systems in C^{n+2}, for some $j = 1, 2, \ldots, n + 1$.

By ordering the coordinates so that $j = n+1$, then $\partial F(q)/\partial x_{n+1} \neq 0$ and $\partial F(q)/\partial y_{n+1} \neq 0$ for all $q \in C^{n+2}$. Suppose further, for $a < b$, that

$$P_{n+1}(C)^b_a = \overline{P_{n+1}(C)^b - P_{n+1}(C)^a} \subset X.$$

Then by the compactness of $f^{-1}(a)$, together with the fact that X is open in $P_{n+1}(C)$, there is an $0 < \epsilon < a$ as $P_{n+1}(C)^b_{a-\epsilon} \subset CX$.

But it is well known that every λ-level surface of f; for $a - \epsilon \leqslant \lambda \leqslant b$, is a compact differentiable $2n+1$-dimensional submanifold of $P_{n+1}(C)$. But since $P_{n+1}(C)$ carries a Riemannian structure (see Brickell and Clark 1970). Thus at each point $p \in p_{n+1}(C)^b_{a-\epsilon}$ there is exactly one such curve passing through each point and cuts λ-level surfaces normally. These curves are given in each coordinate neighbourhood by a system of differential equations:

$$dx_j(t)/dt = \rho_1 g^{jk}(x(t), y(t))\, \partial F'(x(t), y(t))/ox_{k'}$$

$$dy_j(t)/dt = \rho_2 g^{jk}(x(t), y(t))\, \partial F'(x(t), y(t))/oy_{k'}$$

where F' is defined to be $F - m$ ($m = (0, 1)$ which is either a or b) which has the same properties as F and g^{jk} is the reciprocal of the positive definite Riemannian metric g_{ih} on $P_{n+1}(C)$. Here we choose

$$\rho_1 g^{jk} \partial F'(x(t), y(t))/\partial x_j \,.\, \partial F'(x(t), y(t))/\partial x_k = -1$$

$$\rho_2 g^{jk} \partial F'(x(t), y(t))/\partial y_j \,.\, \partial F'(x(t), y(t))/\partial y_k = -1$$

to get parametrizations for which

$$dF'(x(t), y(t))/dt = \partial F'(x(t), y(t))/\partial x_k \,.\, dx_k(t)/dt = -1$$

$$dF'(x(t), y(t))/dt = \partial F'(x(t), y(t))/\partial y_k \,.\, dy_k(t)/dt = -1.$$

Definition 2.5 — Any orthogonal trajectory parametrized in this way is called an ortho-f-arc.

3. The Proofs of the Theorems and the Lemmas

We quote in this section the following two lemmas [see Morsy (1966) for their proofs] which are needed in our proofs of the above theorems and lemmas.

Lemma A — $Q_n(C)$ and $P_{n+1}(R)$ are non-degenerate manifolds of f and they contain all the critical points of f.

Lemma B — Let M be the orbit of $p \in p_{n+1}(C)$ under $SO(n+2)$

(i) If $p \in Q_n(C)$, then $SO(2) \times SO(n)$ is the stabilizer [see Morsy (1966) for the definition of the stabilizer of a point] of p and so M is homeomorphic to $Q_n(C)$.

(ii) If $p \in P_{n+1}(R)$, then $O(1) \times O(n+1)$ is the stabilizer of p and so M is homeomorphic to $P_{n+1}(R)$.

(iii) If $p \in X$, then $SO(n)$ is the stabilizer of p and so M is homeomorphic to $V_{n+2,2}(R)$.

Proof of Theorem 1 — We will show that the triple $(X, V_{n+2,2}, (0, 1))$ is in fact a normal vector bundle of $V_{n+2,2}$ in $P_{n+1}(C)$. Let $N_\epsilon \subset X$ be an ϵ-tube of the compact oriented manifold $V_{n+2,2}(R)_s$ so small that the mapping $P : N_\epsilon \to V_{n+2,2}$ defined by the orthogonal projection of points of N_ϵ onto $V_{n+2,2}$ is well defined, since X has a Riemannian structure induced from $P_{n+1}(C)$, and constitutes fibre decomposition of N_ϵ, with $V_{n+2,2}$ as base space and the orthogonal 1-dimensional cells $F_b(= (0, 1))$ for all $b \in V_{n+2,2}(R)$ as fibres. Thus the quadruple $(N_\epsilon, V_{n+2,2}(R), p, (0, 1))$ is a normal vector bundle of $V_{n+2,2}(R)$ in $P_{n+1}(C)$. By lemma (B), it follows that $X = \bigcup_{\epsilon \in (0,1)} N_\epsilon$, and by the compactness of N_ϵ together with the regularity of f on each fibre (see lemma (A)) we get that the quadruple $(X, V_{n+2,2}, p, (0, 1))$ is a normal vector bundle of $V_{n+2,2}$ in $p_{n+1}(C)$.

Proof of Lemma 1 — The proof of this lemma goes in two steps as follows:

(i) B has a differentiable structure: Choose ϵ small enough and let N_ϵ be an ϵ-tubular neighbourhood of $V_{n+2,2}$ then by Theorem 1, it follows that X is diffeomorphic to $V_{n+2,2} \times (0, 1)$. Since f is regular on X, let γ, γ' be two ortho-f-arcs passing through two distinct points p and q of $V_{n+2,2}.(R)$. Such γ and γ' cuts N_ϵ in a section which is still a product. Hence γ and γ' are diffeomorphic. But this in turn defines a differentiable structure on a neighbourhood of every point of B. Such neighbourhoods constitute bases for B. Therefore B carries a differentiable structure.

(ii) B is diffeomorphic to $(0, 1)$: Since $B = X/SO(n+2)$, i.e., in the sense of Lemma B, it follows that B is a 1-dimensional noncompact connected differentiable manifold. Hence the result. This completes the proof of the lemma.

Proof of Lemma 2 — By Theorem 1 and Lemma 1 we can think of f as the composite mapping: $X \xrightarrow{\pi} B \xrightarrow{g} (0, 1)$, where π is a natural map that sends $b \in X$ into $V_{n+2,2}$ (orbit contains it) i.e., if U is open in X, then $\pi^{-1}(\pi(U)) = SO(n+2)\, U = \bigcup_{h \in SO(n+2)} hU$. Suppose that $f(p) = f(q)$, this implies that $g(a) = g(b)$ if and only if $a = b$. Thus a contradiction.

Proof of Theorem 2 — Since (i) and (ii) are consequences of (iii). Then our proof is devoted to showing (iii). We need only to prove the following:

(a) $V_{n+2,2}(R) \subset$ the level surfaces of f;

(b) $V_{n+2,2}(R) \supset$ the level surfaces of f.

But (a) follows from the following fact, let $p \in P_{n+1}(C)$ and $g \in SO(n + 2)$, then we can see easily that $f(p) = f(p \cdot g)$. It remains only to prove (b) for which we need to prove two things:

(i) $V_{n+2,2}(R)$ and the level surfaces of f have the same dimensions but this is evident, because dim $V_{n+2,2} = 2n + 1$ and the level surfaces of f have codimensions one. This implies that any general orbit is one component of a level surface of f.

(ii) The level surfaces of f are connected, but this follows from Lemma 2. Thus the theorem is proved.

Proof of Theorem 3 — Let N_ϵ be an ϵ-tubular neighbourhood of the compact oriented manifold $Q_n(C)$ so small that the mapping $\pi : N_\epsilon \rightarrow Q_n(C)$ defined by the orthogonal projection of points of N onto $Q_n(C)$ is well defined, since $Q_n(C)$ has a Riemannian structure induced from $P_{n+1}(C)$, which is invariant under the action of $SO(N + 2)$, and constitutes a fibre decomposition of N_ϵ, with $Q_n(C)$ as a base space and the orthogonal 2-planes $F_b(= R^2)$ for all $b \in Q_n(C)$ as fibres. Thus the quadruple $(N_\epsilon, Q_n(C), \pi, R^2)$ is a normal 2-plane of $Q_n(C)$ in $P_{n+1}(C)$. By Theorem 2 we can see easily that ∂N_ϵ (the boundary of N_ϵ) is a union of orbits of $V_{n+2,2}$, but since N_ϵ is connected then ∂N_ϵ is an orbit. By using Theorem 1, it follows that

$$Y = P_{n+1}(C) - P_{n+1}(R) - N_\epsilon$$

is diffeomorphic to $V_{n+2,2} \times (\epsilon, 1)$. Therefore by the identification of the boundaries ∂Y and ∂N_ϵ, $P_{n+1}(C) - P_{n+1}(R)$ is diffeomorphic to the normal 2-plane bundle of $Q_n(C)$ in $P_{n+1}(C)$.

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