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RECONSTRUCTION OF MAXIMAL MINIMALLY NONOUTERPLANAR GRAPHS*

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Ulam's conjecture (1960) is that, a graph G with at least three points can be reconstructed from the family of subgraphs of G obtained by deleting single points of G. In this paper we prove the conjecture for maximal minimally nonouterplanar graphs.

1. INTRODUCTION

The graphs considered are finite, unoriented, without loops or multiple lines. Definitions not given here may be found in Harary (1969). Any graph G determines a collection of subgraphs $G_i = G - v_i$, one for each point v_i of G. Ulam (1960) has conjectured that two graphs with at least three points having pairwise isomorphic collections of such subgraphs must themselves be isomorphic.

A number of cases of Ulam's conjecture have been proved. In particular Manvel (1972) proved it for maximal outerplanar graphs, under the additional restriction that the multiplicities of the isomorphism classes of subgraphs are known. Giles (1974a, b) proved it for outerplanar graphs and for maximal outerplanar graphs, strengthening the theorem of Manvel. We choose to adopt the point of view toward this conjecture which is expressed in Harary (1969).

Conjecture — A graph G with at least three points can be reconstructed uniquely (up to isomorphism) from its collection of subgraphs $G - v_i$.

A set of points of a planar graph G is called an 'inner point set' $s(G)$ of G, if G can be drawn on the plane in such a way that each point of $s(G)$ lies only on the interior and $|s(G)|$ is minimum. Each point of $s(G)$ is said to be an 'inner point' of G. A graph G is said to be 'minimally nonouterplanar' if $|s(G)| = 1$. A minimally nonouterplanar graph G is 'maximal minimally nonouterplanar' if no line can be added without losing minimally nonouterplanarity. A minimally nonouterplanar graph and maximal minimally nonouterplanar graph can be drawn in the plane

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without intersections and so that all but one of their points border the exterior region and we will always assume that maximal minimally nonouterplanar graphs are given in that form. Let C_n be the cycle with n points and P_n the path with n points. The 'wheel' W_n, $n \geqslant 4$ is defined to be the graph $C_{n-1} + K_1$. It is this cycle C_{n-1} which we have in mind when we use the term 'cycle of a wheel'.

In this paper we give the complete solution to the following problem.

Problem (Kulli 1976) — Show that maximal minimally nonouterplanar graphs are reconstructible.

Our proof relies on the version of the theorem for unicyclic graphs, proved by Arora and Gupta (1976) to the effect that a unicyclic graph can be reconstructed from its collection of endpoint deleted subgraphs but for one exception.

2. Preliminaries

Let G be a maximal minimally nonouterplanar graph. Clearly G is nonseparable. Among the regions of G some will have the property that all but one of their lines bound the exterior region of G. Such regions are called extremal. A point of G is called extremal if it is of degree two and if the lines incident with it bound an extremal region. One can easily see that there is a one-to-one correspondence between the extremal regions and extremal points of a maximal minimally nonouterplanar graph G and every point of degree 2 in G is an extremal point.

We now consider a new graph G^w called week dual of G which is the dual graph of G with the point corresponding to the exterior of G deleted. Kulli and Akka (1977) proved that, if G is a nonseparable minimally nonouterplanar graph, then G^w is unicyclic. We state the following simple lemmas.

Lemma 2.1 — A point v is extremal of a maximal minimally nonouterplanar graph G if and only if $(G - v)^w$ is unicyclic and in that case $(G - v)^w$ arises from G^w by deleting the point of G^w corresponding to the region of G having v on its boundary.

In other words, as v ranges over the extremal points of G, $(G - v)^w$ ranges over the endpoint deleted subgraphs of G^w.

Lemma 2.2 — For a maximal minimally nonouterplanar graph G,

$$\triangle\,(G^w) = 2 \text{ if } G \text{ is a wheel,}$$

$$= 3 \text{ otherwise, where } \triangle\,(G^w) \text{ is max deg } G^w.$$

The following are of use.

Theorem A (Kulli 1975a) — Let G be a maximal minimally nonouterplanar graph of order p. Then G has (i) $p - 1$ interior regions and (ii) $2p - 2$ lines.

Theorem B (Kulli 1975b) — Every maximal minimally nonouterplanar graph has exactly one wheel as its subgraph.

Theorem C (Kulli 1975b) — A maximal minimally nonouterplanar graph which is not a wheel, has at least one extremal point.

Theorem D (Arora and Gupta 1976) — If a graph G is unicyclic with $k \geqslant 1$ endpoints, it can be reconstructed from its collection of endpoint deleted subgraphs except when k is equal to two and the two G_i's are isomorphic, each being a line attached to a cycle.

Before the main theorem we prove the following.

Lemma 2.3 — Let G be a maximal minimally nonouterplanar graph with p points and q lines, and let G_i have n_i interior regions, q_i lines. Then

(i) $\deg_G v_i = p - n_i$, (ii) $q = p + q_i - n_i$,

(iii) $\sum_{i=1}^{p} n_i = (p-2)^2$, (iv) $q = \frac{1}{2}(p^2 - \sum_{i=1}^{p} n_i)$.

Proof : Let $\deg_G v_i = d_i$. We know that G is nonseparable and has exactly one inner point. Using these two properties of G it can be easily seen that the removal of v_i from G reduces the number of interior regions by $d_i - 1$. Now by Theorem A, G has $p - 1$ interior regions and $2p - 2$ lines. Hence

$$n_i = (p-1) - (d_i - 1) = p - d_i. \text{ Therefore } d_i = p - n_i.$$

From $n_i = p - d_i$, we have $\sum_{i=1}^{p} n_i = p^2 - 2q = (p-2)^2$. The rest follows trivially.

We see that for a maximal minimally nonouterplanar graph G, Lemma 2.3 enables us to determine p, q and d_i by G_i alone.

Lemma 2.4 — A graph G is a wheel with $n \geqslant 4$ points if and only if one of the G_i's is a cycle C_{n-1} and remaining G_i's are isomorphic to $P_{n-2} + K_1$.

Proof : Suppose G is a wheel with $n \geqslant 4$ points. Then $G = C_{n-1} + K_1$.

Case 1 — Assume $n \geqslant 5$. Then the point v_j of K_1 is the unique inner point of G. Clearly G_j is a cycle C_{n-1}. Now let v_i, $i \neq j$, be any point of G lying on C_{n-1}. The removal of v_i from G reduces C_{n-1} to a path P_{n-2} and hence obviously

$$G_i = P_{n-2} + K_1.$$

Case 2 — Assume $n = 4$. Then it is known that every G_i is isomorphic to C_3. But C_3 can be expressed as $P_2 + K_1$. Hence the conditions are satisfied.

Conversely, suppose one of the G_i's is a cycle C_{n-1} and remaining G_i's are isomorphic to $P_{n-2} + K_1$. It is easy to see that G is of order $n \geqslant 4$. Since G_i's are connected, G is connected. Consider the given G_i which is a cycle C_{n-1}. We know that G is obtained by adding a new point, say v_j, to C_{n-1}. Then G is a wheel if we show that v_j is joind with every point of C_{n-1}. On the contrary suppose v_j is not joined with every point of C_{n-1}. Since G is connected, v_j is joined with at least one of the points of C_{n-1}, say v_k, $k \neq j$. Then $G_k \neq P_{n-2} + K_1$, which is a contradiction.

Corollary 2.4.1 — A maximal minimally nonouterplanar graph G is a wheel if and only if no G_i contains a wheel as a subgraph.

PROOF : The necessity is trivial by Lemma 2.4. To prove the sufficiency, suppose no G_i contains a wheel as a subgraph. Assume G is not a wheel. By Theorem B, G contains a wheel as its subgraph and by Theorem C, G has at least one extremal point, say v_j. Clearly G_j is maximal minimally nonouterplanar. Again by Theorem B, G_j contains a wheel as its subgraph, a contradiction.

One can see that if G_i contains a wheel, say W_n, the unique wheel in G is W_n itself. This enables us to determine the wheel in G from given G_i's when G is not a wheel.

3. THE MAIN THEOREM

Theorem 3.1 — A maximal minimally nonouterplanar graph is reconstructible.

PROOF : Let G be a maximal minimally nonouterplanar graph. By Lemma 2.4, from given G_i's, it is easy to determine whether G is a wheel or not. We consider the following two cases.

Case 1 — Suppose G is a wheel. Choose a G_i which is a cycle say C. To reconstruct G add a new point to C, joining it with every point of C.

Case 2 — Suppose G is not a wheel. By Theorem B, G contains exactly one wheel, say W_n, as its subgraph and by Theorem C, G has at least one extremal point. Also it is clear that if v_i is an extremal point of G, G_i contains a wheel W_n as its subgraph. By (i) of Lemma 2.3, one can choose G_k for which v_k is an extremal point of G. Since then G_k contains a wheel W_n, the wheel in G is determined from G_k. Then label the inner point as v_p. Clearly G_p is a nonseparable outerplanar graph. One can see easily that $G_{ij} = G_{ji}$ where G_{ij} is a graph obtained from G_i by deleting v_j, $i \neq j$. Then G_{kp} is a proper subgraph of G_p. Choose that G_i which is nonseparable, outerplanar, with minimum number of lines and contains G_{kp} as a subgraph. Then G_i with this property is G_p or isomorphic to G_p. Thus from given G_i's, G_p is determined. We now consider two subcases depending on the number of points of W_n.

Subcase 2.1 — Suppose $n \geqslant 5$. Then there is a unique interior region F in G_p which is bounded by a cycle of length $n - 1$. Add a new point to G_p, joining it with every point of F. The graph thus obtained is G.

Subcase 2.2 — Suppose $n = 4$. We make a case distinction based on the nature of G^w. Note that one can reconstruct G^w from the point deleted subgraphs of G but for one exception, taking into account of Lemma 2.1 and Theorem D.

Subcase 2.2.1 — Let G be a graph other than that which is shown in Fig. 1. Then G^w is unicyclic with at least one endpoint such that G^w has no two endpoints with the property that, two endpoint deleted subgraphs are isomorphic, each being a line attached to a cycle. Taking into account of Theorem D, one can reconstruct G^w from extremal point deleted subgraphs of G. We see that G^w has a cycle C_3. Obtain a new graph G^t from G^w by replacing C_3 with a point u, making it incident with the rooted lines at the points on C_3 of G^w. Then obviously G^t is a tree and it is easy to see that G^t and $(G_p)^w$ are isomorphic. Add a new point to G_p in the interior region R corresponding to the point u of G^t, joining it with every point of R. The new graph thus obtained is G.

Subcase 2.2.2 — Let G be graph of Fig. 1. Clearly from given G_i's we get two endpoint deleted subgraphs of G^w which are isomorphic, each being a line attached to a triangle. Arora and Gupta (1976) noted that G^w is not reconstructible by these subgraphs because G^w may be a graph as shown in Fig. 2 or in Fig. 3.

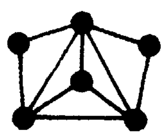

FIG. 1.

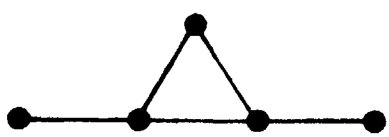

FIG. 2.

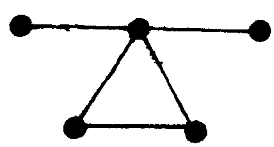

FIG. 3.

Since by Lemma 2.2, $\Delta\ (G^w) = 3$, G^w is not the graph of Fig. 3. Hence G^w is unicycle as shown in Fig. 2. Then to reconstruct G we follow the same procedure as in Subcase 2.2.1.

The following corollary is immediate from Theorem 3.1.

Corollary 3.1.1 — A maximal minimally nonouterplanar graph which is not a wheel is reconstructible from its collection of extremal point deleted subgraphs and an inner point deleted subgraph.

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