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# A NOTE ON TAYLOR'S THEOREM

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In this paper a strong theorem relating to the limit of $\theta \in (0, 1)$ occurring in the Lagrange's form of the remainder in the Taylor's theorem for $f(x)$: $R \to R$ is proved and the same is extended to $f(X) : R^n \to R$. Also certain applications have been brought out.

## 1. INTRODUCTION

In the Taylor's theorem for $f(x) : R \to R$,

$$f(x) = f(a) + (x - a) f'(a) + \dots + \frac{(x - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + R_n,$$

the Lagrange's form of the remainder is

$$R_n = \frac{(x - a)^n}{n!} f^{(n)} (a + \theta(x - a)),\ 0 < \theta < 1. \qquad \dots(1)$$

$\theta$ in (1) depends on the interval, but its limit, as the interval shrinks to zero, is obtained here. This has applications which are considered in the last section.

## 2. EXISTENCE OF THE LIMIT OF $\theta$

If one reviews the problem relating to the limit of $\theta$ (Adikesavan 1974, 1977; Berman 1977; Chatterjea 1970; Hardy 1960) it will be found that no mention is made on the dependence of $\theta$ on $a$, $h$ and $n$, and its uniqueness or otherwise. Indeed $\theta$ is not always uniquely determined. But one can certainly select $\theta$ for specified $a$, $h$ and $n$ such that it does have a limit as $x \to a$ under the assumption that $f(x) \in C^p$, $p$ being a suitable integer greater than $n$. Also, if $h = x - a$ is chosen to be sufficiently small, $\theta$ will be single valued (see the illustrative example in Figs. 1–4). In what follows we will assume that $\theta$ is a unique function with a limit as the interval $h$ shrinks to zero.

$$\text{In (1)}\ \lim_{x \to a} \theta = \frac{1}{n + 1} \quad \text{only when} \quad f^{(n+1)}(a) \neq 0. \qquad \dots(2)$$

For instance, when $n = 1$, $f''(a) = 0$, $f'''(a) \neq 0$, $\lim_{x \to a} \theta = 1/\sqrt{3}$. Now we prove a strong theorem relating to this limit problem for $f(x) : R \to R$.

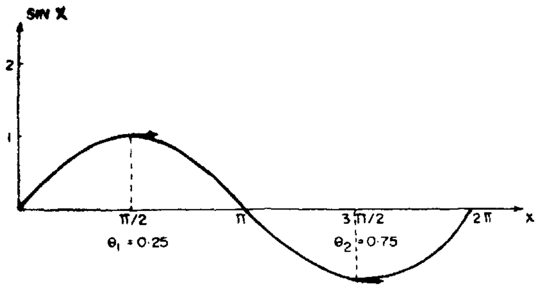


FIG. 1. Plot for $\theta \in (0, 1)$ in the M.V.T., $\sin X = X \cos \theta X$, $X = 2\pi$, $\theta_1 = 0.25$, $\theta_2 = 0.75$.

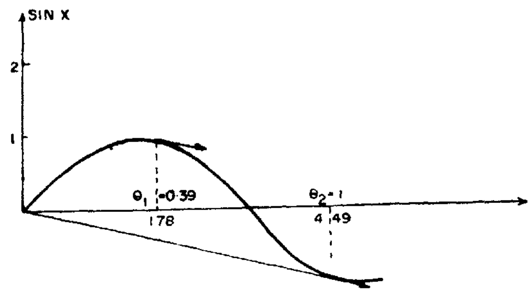


FIG. 2. Plot for $\theta \in (0, 1)$ in the M.V.T., $\sin X = X \cos \theta X$, $X = 4.4934$; $\theta_1 = 0.3887$, $\theta_2 = 1$.

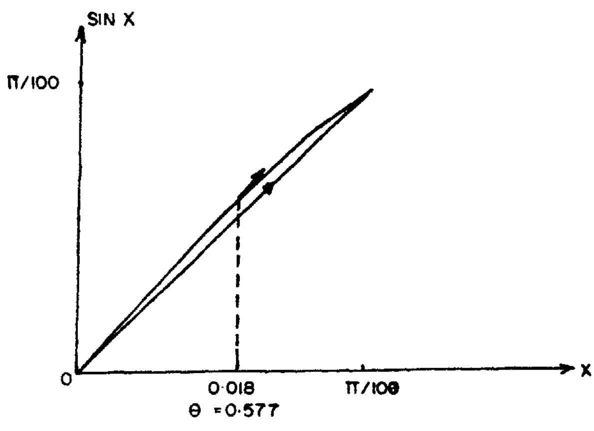


FIG. 3. Plot for $\theta \in (0, 1)$ in the M.V.T., $\sin X = X \cos \theta X$, $X = \pi/100$, $\theta_1 = 0.577$, $\theta_2 = 199.4$.

*Theorem* — If $f(x)$ is differentiable $n + m$ times for $x \in [a, b]$ with $f^{(n+i)}(a) = 0$, $i = 1, 2, \ldots, m - 1$ and $f^{(n+m)}(a) \neq 0$, then in the Taylor's theorem (1),

$$\lim_{x \to a} \theta = \left[\frac{n!\, m!}{(m+n)!}\right]^{1/m}. \qquad \ldots(3)$$

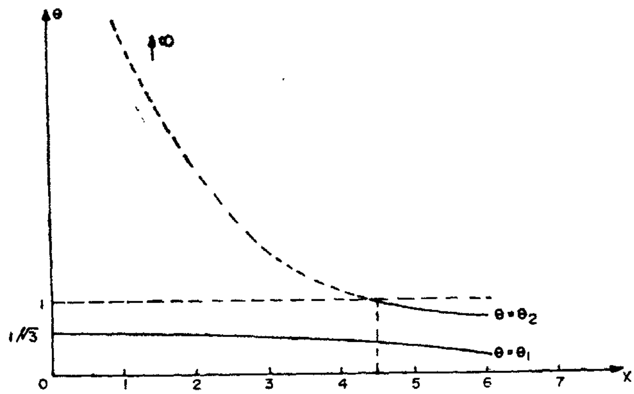


FIG. 4. $X - \theta$ plot for $\sin X = X \cos \theta X$, $X \in (0, 2\pi)$ and $\theta \in (0, 1)$.

PROOF : Upon using Taylor's theorem, $\forall\ x \in (a, b]\ \exists\, \theta_1 \in (0, 1) \ni$

$$f(x) = f(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{m+n}}{(m+n)!} f^{(m+n)}(a + \theta_1(x-a)). \qquad \ldots(4)$$

Also, $\exists\ \theta_2 \in (0, 1) \ni$

$$f^{(n)}(a + \theta(x-a)) = f^{(n)}(a) + \frac{\theta^m(x-a)^m}{m!} f^{(n+m)}(a + \theta_2\theta(x-a)). \qquad \ldots(5)$$

From (1), (4) and (5),

$$\theta^m f^{(n+m)}(a + \theta_2\theta(x-a)) = \frac{n!\, m!}{(n+m)!} f^{(n+m)}(a + \theta_1(x-a)).$$

It follows that, as $f^{(n+m)}(a) \neq 0$,

$$\lim_{x \to a} \theta = \left[\frac{n!\, m!}{(n+m)!}\right]^{1/m}.$$

*Corollary* 1 — It is interesting to observe that, if $f(x) = x^{n+p}$, $p = 1, 2, 3, \ldots$, then, with $a = 0$,

$$\theta = \left[\frac{n!\, p!}{(n+p)!}\right]^{1/p}.$$

Further, if $f(x) = \sum_{i=1} a_i x^{n+p+i}$, $p = 1, 2, \ldots$ with $a_1 \neq 0$,

$$\lim_{x \to a} \theta = \left[\frac{n!\, p!}{(n+p)!}\right]^{1/p}.$$

For instance, in the first mean value theorem ($n = 1$) viz.

$$f(x) = f(0) + xf'(\theta x), \text{ if}$$

$$f(x) = x^4, \ \theta = 4^{-1/3}$$

$$= x^4 + x^5, \ \lim_{x\to 0} \theta = 4^{-1/3}$$

$$= x^4 + x^2, \ \lim_{x\to 0} \theta = \tfrac{1}{2}.$$

*Corollary* 2 — It can be proved that in the Taylor's theorem for $f(X) : R^n \to R$ viz. (Goffman 1965, Graves 1956, Lang 1969, Seeley 1970)

$$f(X + H) = f(X) + \sum_{i=1}^{m-1} \frac{1}{i!} (H \cdot \nabla)^i f(X) + \frac{1}{m!} (H \cdot \nabla)^m f(X + \theta H),$$

$$0 < \theta < 1 \qquad \text{...(6)}$$

$$\lim_{|H|\to 0} \theta = \left[\frac{m!\,p!}{(m+p)!}\right]^{1/p} \qquad \text{...(7)}$$

if $(H \cdot \nabla)^{m+i} f(X) = 0$, $i = 1, 2, ..., p - 1$ and $(H \cdot \nabla)^{m+p} f(X) \neq 0$.

## 3. Applications

The following results have been obtained using the notion of "lim θ" in (6).

1. If $f(tX) = t^3 f(X)$ i.e. if $f(X)$ is a homogeneous polynomial (Seeley 1970) of degree three, $f(X) = (X \cdot \nabla) f(X/\sqrt{3})$.

2. If $f(tX) = t^4 f(X)$, $f(X) = (X \cdot \nabla) f(X/4^{1/3}) = \frac{1}{2}(X \cdot \nabla)^2 f(X/\sqrt{6})$.

3. If $f(tX) = t^5 f(X)$, $f(X) = (X \cdot \nabla) f(X/5^{1/4}) = \frac{1}{2}(X \cdot \nabla)^2 f(X/10^{1/3})$

$$= \tfrac{1}{6}(X \cdot \nabla)^3 f(X/\sqrt{10}).$$

A generalization of the above results is as follows.

4. If $f(tX) = t^N f(X)$, where $t$ is a scalar and $N$ an integer $\geqslant 3$,

$$f(X) = \frac{1}{k!} (X \cdot \nabla)^k f(X/p_k),$$

where

$$p_k = [{}_N C_k]^{1/(N-k)}, \ k = 1, 2, ..., N - 2. \qquad \text{...(8)}$$

5. Euler's theorem on homogeneous functions viz. $(X \cdot \nabla) f(X) = Nf(X)$ can be obtained from (8). Choose $k = 1$, then (8) becomes $f(X) = (X \cdot \nabla) f(X/p_1)$. Now we make a simple transformation (scaling) $X \leftarrow p_1 X$ and obtain the Euler's theorem.

The following results will exemplify that the notion of "lim $\theta$" is propitious for application in Numerical Analysis. The unknown point $\xi$ in the remainder term $\frac{h^n}{n!} f^{(n)}(\xi)$ becomes "nearly identifiable" for sufficiently small $h$ with the adoption $\xi(\theta) \leftarrow \xi$ (lim $\theta$). With this adoption some well known formulas can be easily obtained. We now add to the list above:

6. $f(a + h) = f(a) + hf'(a + \frac{1}{2}h) + O(h^3)$. This is the well known numerical differentiation formula (Hildebrand 1956), $y'_k \approx \frac{1}{h} \mu\delta y_k$, where $\mu$ is the averaging operator and $\delta$, the central difference operator.

7. If $f(x)$ is odd, $f(x) = xf'(x/\sqrt{3}) + O(x^5)$. This can be modified as

$$\int_{-1}^{1} f(x)\, dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

which is simply the 2-point Gauss-Legendre formula (Demidovich and Moron 1973) in numerical integration.

8. In the Simpson rule,

$$\int_0^{2h} y\, dx = \frac{h}{3}(y(0) + 4y(h) + y(2h)) - \frac{h^5}{90} y^{(4)}(\xi),$$

upon setting $\xi = 2\theta h$, $\lim_{h\to 0} \theta = \frac{1}{2}$, if $y^{(5)}(0) \neq 0$. An error estimate (not bounds) can be obtained as $-\frac{h^5}{90} y^{(4)}(h)$ with the adoption $\xi(\theta) \leftarrow \xi$ (lim $\theta$). For example, consider $\int_0^1 (1 + x)^{-1}\, dx$. Simpson rule gives $\log_e 2$ as 0.69444 ... against 0.693147 ... with an error of $-1.3 \times 10^{-3}$. Our error estimate is $-1.1 \times 10^{-3}$. The error bounds are $(-8.3 \times 10^{-3}, -2.6 \times 10^{-4})$.

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