Indian J. pure appl. Math., 11(4) : 450-456, April 1980

SOME EXAMPLES OF NON NORMAL OPERATORS

J. D. ACHARYA

Department of Mathematics, Gujarat University, Ahmedabad 380009

(*Received* 6 *April* 1979)

Sufficient conditions for an operator $T = A \oplus B$ to be of class R, Class G_1, Class H_1, spectraloid and transloid are obtained and some properties of class R and class G_1 are discussed.

INTRODUCTION

If T is an operator (a bounded linear transformation) on the complex Hilbert space H, then

$$T \in G_1 \text{ if } \| (T - uI)^{-1} \| = \frac{1}{d(u, \sigma(T))}, \; u \notin \sigma(T)$$

$$T \in R \text{ if } \| (T - uI)^{-1} \| = \frac{1}{d(u, \overline{W(T)})}, \; u \notin \overline{W}(T)$$

$$T \in H_1 \text{ if } \| (T - uI)^{-1} \| = \frac{1}{d(u, \tilde{\sigma}(T))}, \; u \notin \tilde{\sigma}(T)$$

where $\sigma(T)$, $\overline{W}(T)$ and $\tilde{\sigma}(T)$ are respectively the spectrum, the closure of numerical range and the hen-spectrum of T. According to Fujii (1973a) the hen-spectrum of T is the complement of the unbounded component of the complement of $\sigma(T)$, i.e.

$$\tilde{\sigma}(T) = [[\sigma(T)^c]_\infty]^c$$

and $\sigma(T) \subset \tilde{\sigma}(T) \subset \text{con } \sigma(T) \subset \overline{W}(T)$ where con $\sigma(T)$ denotes the convex hull of $\sigma(T)$. In the same paper, a beautiful characterization of operators of class R is given as follows

$$T \in R \text{ if and only if } \overline{W}(T) = \tilde{\sigma}(T).$$

$r(T)$, $w(T)$ and $\| T \|$ are respectively the spectral radius, numerical radius and norm of T (Halmos 1978).

An operator T is normaloid if $w(T) = \| T \|$ or equivalently $r(T) = \| T \|$, transloid if $T - \lambda$ is normaloid for all complex numbers λ and spectraloid if $r(T) = w(T)$ (Halmos 1978).

A closed set S in complex plane is a spectral set for T if $\sigma(T) \subset S$ and

$$\| f(T) \| \leqslant \sup_{z \in S} | f(z) |$$

where f is any rational function with poles off S (Lebow 1963).

An operator T is spectroid, hen-spectroid and numeroid according as $\sigma(T)$, $\tilde{\sigma}(T)$ and $\bar{W}(T)$ is a spectral set for T (Fujii 1973b).

In section I, we obtain sufficient conditions for an operator to be in class R, class G_1 and class H_1, generalizing the respective results of Furuta (1977), Luecke (1972b) and Fujii (1973a) and obtain similar results for spectraloids and transloids.

In section II, we discuss some properties of class R and class G_1.

SECTION I

Luecke (1972a) proved the following theorem.

Theorem A — If A is an operator on H, then $A \oplus N \in R$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supseteq \partial W(A)$.

Furuta (1977) has given a counter-example to show that the conditions are not sufficient.

We give below a general method to construct operators of class R.

Theorem 1 — If A be any operator and $B \in R$ such that $\bar{W}(A) \subset \bar{W}(B)$, then $T = A \oplus B \in R$.

PROOF : We have $\bar{W}(B) \subset \bar{W}(T)$.

But $$\bar{W}(T) = \text{Con}\,(W(A)\; U W(B)) \subset \text{Con}\,(\bar{W}(A)\; U \bar{W}(B)) = \text{Con}\;\bar{W}(B) = \bar{W}(B).$$

Thus $$\bar{W}(T) = \bar{W}(B).$$

Since $\bar{W}(A) \subset \bar{W}(B)$ for $u \notin \bar{W}(T) = \bar{W}(B)$
$$d(u, W(A)) \geqslant d(u, W(B)).$$

Now for $u \notin \bar{W}(T)$
$$\| (T - uI)^{-1} \| = \max \{ \| (A - uI)^{-1} \|, \| (B - uI)^{-1} \| \}$$
$$= \max \left\{ \frac{1}{d(u, W(A))}, \frac{1}{d(u, W(B))} \right\}$$
$$= \frac{1}{d(u, W(B))}$$
$$= \frac{1}{d(u, W(T))}.$$

Therefore $T \in R$.

A method to construct operators of class G_1 is given in Luecke (1972b). Here we give a more general method for the same.

Theorem 2 — If A is any operator and $B \in G_1$ with $\bar{W}(A) \subset \sigma(B)$, then $T = A \oplus B \in G_1$.

PROOF : We have $\sigma(T) = \sigma(A) \, U \sigma(B) = \sigma(B)$. For $u \notin \sigma(T)$,

$$\| (T - uI)^{-1} \| = \max \{ \| (A - uI)^{-1} \|, \| (B - uI)^{-1} \| \}$$

$$= \max \left\{ \frac{1}{d(u, W(A))}, \frac{1}{d(u, \sigma(B))} \right\}$$

$$= \frac{1}{d(u, \sigma(B))}$$

$$= \frac{1}{d(u, \sigma(T))}.$$

Therefore $T \in G_1$.

Remarks : In Theorem 2, the condition $\bar{W}(A) \subset \sigma(B)$ cannot be weakened as $\bar{W}(A) \subset \tilde{\sigma}(B)$. Since, for $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and B, a bilateral shift operator, we have $\bar{W}(A) \subset \tilde{\sigma}(B)$ but $T = A \oplus B \notin G_1$, Fujii (1973a).

Fujii (1973a) has given a method to construct operators of class H_1 as follows.

Theorem B — If A is any operator and B is a normal operator with $\bar{W}(A) \subset \tilde{\sigma}(B)$, then $T = A \oplus B \in H_1$.

We generalize the same as follows.

Theorem 3 — If A is any operator and $B \in H_1$ with $\bar{W}(A) \subset \tilde{\sigma}(B)$, then $T = A \oplus B \in H_1$.

PROOF : Since $\tilde{\sigma}(A) \subset \bar{W}(A) \subset \tilde{\sigma}(B)$ for $u \notin \tilde{\sigma}(T) = \tilde{\sigma}(B)$,

$$\| (T - uI)^{-1} \| = \max \{ \| (A - uI)^{-1} \|, \| (B - uI)^{-1} \| \}$$

$$= \max \left\{ \frac{1}{d(u, W(A))}, \frac{1}{d(u, \tilde{\sigma}(B))} \right\}$$

$$= \frac{1}{d(u, \tilde{\sigma}(B))}$$

$$= \frac{1}{d(u, \tilde{\sigma}(T))}.$$

Therefore $T \in H_1$.

Being inspired by Luecke (1972a, b) and Fujii (1971; 1973a, b) we give a method to construct spectraloid operators as follows.

Theorem 4 — If A be any operator and B be a spectraloid operator such that $w(A) \leqslant r(B)$, then $T = A \oplus B$ is spectraloid.

PROOF : Since B is spectraloid, $r(B) = w(B)$. Now

$$r(T) = \max \{r(A), r(B)\} = r(B),$$

since $r(A) \leqslant w(A)$ and $w(A) \leqslant r(B)$ and $w(T) = \max \{w(A), w(B)\} = w(B) = r(B)$ because $w(A) \leqslant r(B) \leqslant w(B)$. Thus $r(T) = w(T)$ i.e. T is spectraloid.

Corollary 1 — If A be any operator and B be a normal operator with

$$w(A) \leqslant r(B),$$

then $T = A \oplus B$ is spectraloid.

Fujii (1971) has given a method to construct normaloid operators.

Theorem C — If A is any operator and B is normaloid such that $\| A \| \leqslant \| B \|$, then $T = A \oplus B$ is normaloid.

We remark that the condition $\| A \| \leqslant \| B \|$ cannot be weakened as

$$r(A) \leqslant \| B \| \quad \text{or} \quad w(A) \leqslant \| B \|.$$

Since for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and B a normal operator with $\sigma(B) = \bar{W}(A)$, both the above conditions hold but $T = A \oplus B$ is not normaloid (Luecke 1972a).

Using the above theorem, we give a method to construct transloid operators.

Theorem 5 — If A be any operator and B be a transloid with

$$\| (A - \lambda I) \| \leqslant \| (B - \lambda I) \|$$

for all complex numbers λ, then $T = A \oplus B$ is a transloid.

PROOF : Since $T - \lambda I = (A - \lambda I) \oplus (B - \lambda I)$ and $B - \lambda I$ is normaloid with $\| A - \lambda I \| \leqslant \| B - \lambda I \|$. $T - \lambda I$ is normaloid by Theorem C.

Remarks : Theorems 1, 2, 3, 4 are respectively useful to obtain sufficient conditions for an operator to be essentially R, essentially G_1, essentially H_1 and essentially spectraloid.

SECTION II

It is known that class G_1 and class R are proper sub-sets of class H_1 (Fujii 1973a).

In the following theorem, we show that the union of these two classes is also a proper sub-set of H_1.

Theorem 6 — $G_1 \cup R$ is a proper sub-set of class H_1.

Proof : Let $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, then $\sigma(A) = \{-1\}$ and $\bar{W}(A) =$ unit disc about $\lambda = -1$. Suppose B is a normal operator with

$$\sigma(B) = \{\lambda \mid |\lambda + 1| = 1\} \cup \{\lambda \mid |\lambda| = 3, \text{Re}\,\lambda \leqslant 0\}.$$

Define $T = A \oplus B$.

Now $\sigma(T) = \sigma(B)\, U \{-1\}$ and

$$\tilde{\sigma}(T) = \tilde{\sigma}(B) = \{\lambda \mid |\lambda + 1| \leqslant 1\} \cup \{\lambda \mid |\lambda| = 3, \text{Re}\,\lambda \leqslant 0\}.$$

As $\bar{W}(A) \subset \tilde{\sigma}(B)$, $T \in H_1$ by theorem B and $\tilde{\sigma}(T) \neq \bar{W}(T)$. Therefore $T \notin R$.

Also $\| (A + \frac{1}{2})^{-1} \| = 2 \left\| \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \right\|$ for $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\| (A + \frac{1}{2})^{-1} x \| > 2$.

Hence $T \notin G_1$ also because if $T \in G_1$

$$2 < \| (A + \tfrac{1}{2})^{-1} x \| \leqslant \| (A + \tfrac{1}{2})^{-1} \| \leqslant \| (T + \tfrac{1}{2})^{-1} \|$$

$$= \frac{1}{d(-\frac{1}{2}, \sigma(T))} = 2$$

a contradiction.

It is known that for hyponormal operators

$$E(T) \cap W(T) \subset \sigma_p(T) \qquad \ldots(*)$$

where $E(T)$ denote the extreme points of $\bar{W}(T)$ and $\sigma_p(T)$ denote the point spectrum of T, Saitô (1971) has given an example to show that convexoid operators do not satisfy (*). Here we give an example to show that (*) does not hold even for operators of class G_1 and R. The problem is still open for a transloid operator.

Theorem 7 — There exists an operator in $G_1 \cap R$ such that

$$E(T) \cap W(T) \subset \sigma_p(T)$$

does not hold.

Proof : Let $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ and B be a normal operator with

$$\sigma(B) = \{\lambda \mid |\lambda| \leqslant 1\} \text{ and } B\partial\sigma(B) \cap \sigma_p(B) = \phi.$$

For the operator $T = A \oplus B$

$$\sigma(A) = \{\tfrac{1}{2}\}, \sigma(T) = \sigma(B) = \text{con}\, \sigma(B) = \text{con}\, \sigma(T).$$

$$\bar{W}(A) = W(A) = \{\lambda \mid |\lambda - \tfrac{1}{2}| \leqslant \tfrac{1}{2}\}$$

$$\bar{W}(T) = \text{con } \sigma(B) = \text{con } \sigma(T)$$

by our construction $1 \notin \sigma_p(B)$ i.e. $1 \notin \sigma_p(T)$. But $1 \in E(T) \cap W(T)$.

Further $\bar{W}(A) \subset \sigma(B)$ implies $T \in G_1$ by Theorem 2 and $\bar{W}(T) = \tilde{\sigma}(T)$ implies $T \in R$. Thus $T \in G_1 \cap R$.

Theorem 8 — There exists a transloid in R which is not in G_1 and vice versa.

PROOF : Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $\sigma(A) = \{0\}$, $\bar{W}(A) = \frac{D}{2}$ where D is the unit disc. Let B be a normal operator with $\sigma(B) = C$ where C is unit circle in complex plane. For $T = A \oplus B$ it is easy to verify that $T \notin G_1$ and $T \in R$ since

$$\bar{W}(T) = \tilde{\sigma}(T) = D.$$

Now since $\| A + \lambda \| = \sqrt{\frac{2p + 1 + \sqrt{4p + 1}}{2}}$ where $p = |\lambda|^2$ and

$$\| B + \lambda \| = |\lambda| + 1.$$

It is easy to show that $\| A + \lambda \| \leqslant \| B + \lambda \|$, hence T is transloid by Theorem 5.

For the converse we can consider finite dimensional normal operator with more than one point in $\sigma(T)$ and dim $H \leqslant 2$.

Remarks : There exists a numeroid in R which is not in G_1 and vice versa.

In the above example, A is completely non normal with $\| A \| = 1$ hence unit disc is a spectral set for A (Saitô 1971). Since $S \cup \bar{W}(A) \subset \bar{W}(B)$ where S is spectral set for A, by Fujii (1971) T is a numeroid.

Finally we give some results about spectroids, hen-spectroids (Fujii 1973b) and class R. It is known that spectroids are contained in class G_1 (Lebow 1963).

Theorem 9 — There is an operator in R which is not spectroid (hen-spectroid) and vice-versa.

PROOF : Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and B be a normal operator with $\sigma(B) = D$, where D is the unit disc in the complex plane.

Clearly $T = A \oplus B \in R$.

Here $\sigma(T) = D$ is not the spectral set for T since for $f(z) = z$,

$$\begin{aligned} \| f(T) \| = \| f(A \oplus B) \| &= \| f(A) \oplus f(B) \| \\ &= \max \{ \| f(A) \|, \| f(B) \| \} \\ &= 2. \end{aligned}$$

But $\sup | f(z) |_{z \in D} = 1$.

Therefore $\| f(T) \| > \sup | f(z) |_{z \in D}$, hence T is not spectroid.

For the above example $\sigma(T) = \bar{\sigma}(T)$ hence, same $T \in R$ is not a hen-spectroid.

For the converse we can consider any finite dimensional normal operator with more than one point in the spectrum.

Acknowledgement

The author is thankful to the U.G.C. for the award of teacher fellowship and is indebted to Dr I. H. Sheth for his valuable guidance, kind help and encouragement in the preparation of this article.

References

Fujii, M. (1971). On some examples of non-normal operators—I. *Proc. Japan. Acad.*, **47**, 458–63.

——— (1973a). On some examples of non-normal operators—II. *Proc. Japan Acad.*, **49**, 118–23.

——— (1973b). On some examples of non-normal operators—III. *Proc. Japan Acad.*, **49**, 124–29.

Furuta, T. (1977). Relations between generalized growth conditions and several classes of convexoid operators. *Can. J. Math.*, **29**, 1010–30.

Halmos, P. R. (1978). Hilbert Space Problem Book. D. Van Nostrand, Princeton.

Lebow, A. (1963). On Von Neumann's theory of spectral sets. *J. Math. Analysis, Applic.*, **7**, 64–90.

Luecke, G. R. (1972a). A class of operators on Hilbert space. *Pacific J. Math.*, **41**, 153–56.

——— (1972b). Topological properties of operators satisfying condition G_1 on Hilbert space. *Trans. Am. math. Soc.*, **172**, 35–43.

Saitô, T. (1971). Lecture Notes on Operator Theory. Tohoku University, Japan.