Indian J. pure appl. Math., **11**(4) : 457–461, April 1980

SUBCLASSES OF CONVEXOID OPERATORS II

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(*Received* 14 *May* 1979)

Using the notation of operator radii $w_\rho(.)$, we define new classes S_ρ, for each $\rho \geqslant 1$. An operator $T \in S_\rho(\rho \geqslant 1)$ if $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$, for all $z \notin \bar{W}(T)$. It is shown that, class S_ρ is a subset of class of convexoid operators and there exists a non-normal operator on a finite dimensional Hilbert space H, in this class.

Introduction

Let $B(H)$ denote the set of all bounded linear transformations on a complex Hilbert space H. Let $\sigma(T)$, con $\sigma(T)$ and $\bar{W}(T)$ respectively denote the spectrum, convex-hull of $\sigma(T)$ and the closure of numerical range of T. According to Fujii (1973) hen-spectrum of T (denoted by $\tilde{\sigma}(T)$) is the complement, of the unbounded component, of the complement of $\sigma(T)$, i.e. $\tilde{\sigma}(T) = [[(\sigma(T))^c]_\infty]^c$;

$$\sigma(T) \subseteq \tilde{\sigma}(T) \subseteq \text{con } \sigma(T) \subseteq \bar{W}(T) \text{ and } \partial\tilde{\sigma}(T) \subseteq \sigma(T),$$

where ∂M denotes the boundary of set M. Clearly for $z \notin \tilde{\sigma}(T)$,

$$d(z, \sigma(T)) = d(z, \tilde{\sigma}(T)).$$

Let $r(T)$, $w(T)$ and $\| T \|$ denote the spectral radius, the numerical radius and the norm of T. An operator T is normaloid if $w(T) = \| T \|$, spectraloid if $r(T) = w(T)$ and convexoid if con $\sigma(T) = \bar{W}(T)$.

$$T \in G_1 \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \sigma(T)), \text{ for all } z \notin \sigma(T)$$

$$T \in H_1 \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \tilde{\sigma}(T)), \text{ for all } z \notin \tilde{\sigma}(T)$$

$$T \in S \text{ if } \| (T - zI)^{-1} \| = 1/d(z, \sigma(T)), \text{ for all } z \notin \bar{W}(T).$$

Let $C_\rho(\rho > 0)$ be the class of all operators with unitary ρ-dilations (Sz-Nagy and Foias 1970). According to Holbrook (1968), an operator radius $w_\rho(T)$ is defined as

$$w_\rho(T) = \inf \{u : u > 0 \text{ and } u^{-1}T \in C_\rho\},$$

in particular $w_1(T) = \| T \|$ and $w_2(T) = w(T)$. Further $w_\rho(.)$ is homogeneous, i.e. $w_\rho(zT) = | z | \, w_\rho(T)$ for all complex numbers z.

$T \in B(H)$ is an operator of class $M_\rho (\rho \geqslant 1)$ if, for all $z \notin \sigma(T)$,

$$w_\rho[(T - zI)^{-1}] = 1/d(z, \sigma(T)) \text{ (Patel 1976).}$$

T is an operator of class $H_\rho(\rho \geqslant 1)$, if $w_\rho((T - zI)^{-1}) = 1/d(z, \tilde{\sigma}(T))$, for all $z \notin \tilde{\sigma}(T)$ (Acharya 1980). It is further known that $M_1 = G_1$ (Patel 1976) and for $\rho \geqslant 1$, $M_\rho \subseteq H_\rho$; H_ρ is a proper subset of convexoid operators (Acharya 1980).

We define, classes S_ρ for each $\rho \geqslant 1$, as follows :

Definition : Let $\rho \geqslant 1$. $T \in S_\rho$ if $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$ for all $z \notin \bar{W}(T)$.

It is clear that,

(i) For $\rho = 1$, $w_1(T) = \| T \|$. Therefore class S_ρ coincides with class S, defined by Furuta (1977). In this case $(T - zI)^{-1}$ is normaloid for each $z \notin \bar{W}(T)$.

(ii) For $\rho = 2$, $w_\rho((T - zI)^{-1}) = w((T - zI)^{-1}) = r((T - zI)^{-1})$ for all $z \notin \bar{W}(T)$. Hence, $(T - zI)^{-1}$ is spectraloid for $z \notin \bar{W}(T)$.

(iii) For $\rho \geqslant 1$, $w_\rho((T - zI)^{-1}) = r((T - zI)^{-1})$, for all $z \notin \bar{W}(T)$ implies that $(T - zI)^{-1}$ is ρ-oid for all $z \notin \bar{W}(T)$.

(iv) Since $w_\rho(.)$ is non-increasing as ρ increases, we have $S_\rho \subseteq S_{\rho'}$ whenever $\rho \leqslant \rho'$.

Theorem 1 — (i) Class S_ρ is arcwise connected.

(ii) Class S_ρ is translation invarient.

PROOF : We have $\sigma(\alpha T + \beta) = \alpha\sigma(T) + \beta$ and $W(\alpha T + \beta) = \alpha W(T) + \beta$ for complex numbers α and β. Further it is shown in (Holbrook 1968) that

$$w_\rho(\alpha T) = | \alpha | \, w_\rho(T).$$

Now it is easy to see that $T \in S_\rho$ implies that $\alpha T \in S_\rho$ for every complex number α. Hence the ray in $B(H)$ through T is contained in S_ρ. Therefore S_ρ is arcwise connected.

Since, $\sigma(T)$ and $W(T)$ are translation invarients, it is easy to see that class S_ρ is also translation invarient.

It is known that H_1 is a subset of S (Furuta 1977) i.e. $H_\rho \subseteq S_\rho$ is true for $\rho = 1$. We generalize the same for each $\rho \geqslant 1$.

Theorem 2 — For each $\rho \geqslant 1$, class H_ρ is contained in class S_ρ.

PROOF : If $T \in H_\rho$, then $w_\rho((T - zI)^{-1}) = 1/d(z, \tilde{\sigma}(T))$ for all $z \notin \tilde{\sigma}(T)$. Now for $z \notin \tilde{\sigma}(T)$, $d(z, \sigma(T)) = d(z, \tilde{\sigma}(T))$. $\sigma(T) \subseteq \bar{W}(T)$. Hence for $z \notin \bar{W}(T)$,

$$w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T)) \text{ or } T \in S_\rho.$$

Therefore, we have $H_\rho \subseteq S_\rho$.

In our next result, we show that above inclusion is proper for $1 \leqslant \rho \leqslant 2$. We require Theorem 3.2 (Furuta 1977).

Theorem A (Furuta 1977) — If A is an operator and $B \in S$ such that

$$d(z, \sigma(B)) \leqslant d(z, W(A))$$

for all $z \notin \bar{W}(B)$ then $T = A \oplus B \in S$.

Example 1 — Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and B be a normal operator with

$$\sigma(B) = \{z : | z - 3 | = 4, \text{Re } z \leqslant 3\} \, U \, \{z : | z - 3 | = 2, \text{Re } z \leqslant 3\}.$$

Consider $T = A \oplus B$. Since $d(z, \sigma(B)) \leqslant d(z, W(A))$ for all $z \notin \bar{W}(B)$ and B is normal (hence belonging to S), by Theorem A, $T \in S = S_1 \subseteq S_\rho$. Now

$$w((T - zI)^{-1}) = \max \{w((A - zI)^{-1}), w((B - zI)^{-1})\},$$

$$w((A - zI)^{-1}) = (| z | + 1)/ | z |^2 \text{ and } w((B - zI)^{-1}) = 1/d(z, \sigma(B)).$$

For $z = 1/10 \notin \tilde{\sigma}(T) = \{0\} \, U\sigma(B)$, $w((T - zI)^{-1}) = 110$ and $1/d(z, \sigma(T)) = 10$. Hence $T \notin H_2$, i.e. $T \notin H_\rho$ for $1 \leqslant \rho \leqslant 2$.

To show that the operators in class S_ρ are convexoid operators, we require the following generalization of Theorem 3 (Patel 1976).

Theorem B (Furuta 1978) — T is convexoid if and only if

$$w_\rho((T - zI)^{-1}) \leqslant 1/d(z, \text{con } \sigma(T))$$

for all z whose absolute values are sufficiently large.

Theorem 3 — Class S_ρ is a subset of class of convexoid operators.

PROOF : Let $T \in S_\rho$. Hence $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$ for all $z \notin \bar{W}(T)$. Now $\sigma(T) \subseteq \text{con } \sigma(T)$ and $\bar{W}(T)$ is compact, implies that

$$w_\rho((T - zI)^{-1}) \leqslant 1/d(z, \text{con } \sigma(T)),$$

for all z whose absolute values are sufficiently large. Hence T is convexoid by Theorem B.

We give an example to show that the above inclusion is proper for $1 \leqslant \rho \leqslant 2$.

Example 2 — Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and B be a diagonal operator with

$$\text{dig } B = \{\sqrt{2}, -\sqrt{2}, \sqrt{2i}, -\sqrt{2i}\}.$$

Consider $T = A \oplus B$. Since $\bar{W}(A) \subseteq \bar{W}(B)$ and B is covexoid, by Fujii (1971), T is convexoid. Now we show that $T \notin S_2$. We have

$$w((T - zI)^{-1}) = \max \{w((A - zI)^{-1}), w((B - zI)^{-1})\},$$
$$w((A - zI)^{-1}) = (1 + |z|)/|z|^2, \; w((B - zI)^{-1}) = 1/d(z, \sigma(B)).$$

For $z = 1 + i \notin \bar{W}(T)$, $w((T - zI)^{-1}) = (1 + \sqrt{2})/2$

and $\quad 1/d(z, \sigma(T)) = (2 + \sqrt{2})/4.$

Hence $T \notin S_2$. i.e. $T \notin S_\rho$ for $1 \leqslant \rho \leqslant 2$.

It is known that, if $\dim H < \infty$ or $\sigma(T)$ is finite, then $T \in H_\rho$ implies that T is normal (Acharya 1980). It is interesting to know that this property does not hold for class S_ρ.

Theorem 4 — There exists a non-normal operator (on finite dimensional Hilbert space H) belonging to class S_ρ, even if $\dim H < \infty$.

PROOF : Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and B be a diagonal operator with

$$\text{dig } B = \{z_i : z_i = \tfrac{7}{2} e^{2\pi/i}, i = 1, 2, \ldots, 21.\}$$

Consider $T = A \oplus B$. Clearly T is an operator on a finite dimensional Hilbert space H. Since A is non-normal, T is also non-normal. Here $d(z, \sigma(B)) \leqslant d(z, W(A))$, for all $z \notin \bar{W}(B)$, hence by Theorem A, $T \in S = S_1 \subseteq S_\rho$.

It is shown (Patel and Gupta 1975) that if $T \in M_\rho$ and $\sigma(T) \subseteq C$, where C is the unit circle in the complex plane, then T is unitary. It is shown (Acharya 1980) that if $\sigma(T)$ is a proper subset of C and $T \in H_\rho$ then T is unitary. Here we show that :

Theorem 5 — Let $T \in S_\rho$. If $\sigma(T) \subseteq \{z : z = e^{i\theta}, \alpha < \theta < \alpha + \pi\}$, then T is unitary.

PROOF : Since $T \in S_\rho$, and $\sigma(T)$ is as defined above, we have

$$w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T)) \leqslant 1/(\,|\,|z| - 1\,|\,),$$

for all $z \notin \bar{W}(T)$. In particular we have $w_\rho(T^{-1}) \leqslant 1$

and $\quad w_\rho((T - zI)^{-1}) \leqslant 1/(|z| - 1)$

for $|z| > 1$. Therefore the result follows from Theorem 2 and Corollary 1 of Patel and Gupta (1975).

Lastly we show that the class S_ρ is not closed under multiplication and inversion.

Theorem 6 — There exists a non-singular operator T contained in class S_ρ such that (i) $T^2 \notin S_\rho$ for any $\rho \geqslant 1$, (ii) $T^{-1} \notin S_\rho$ for any $\rho \geqslant 1$.

PROOF : Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and B be a normal operator with $\sigma(B) = W(A)$. Consider $T = A \oplus B$. Here $T \in G_1$ by (Luecke 1972). Now

$$G_1 = M_1 \subseteq M_\rho \subseteq H_\rho \subseteq S_\rho$$

implies that $T \in S_\rho$. Further it is shown Luecke (1972) that T^2 is not convexoid. It is further proved by Patel and Gupta (1975) that T^{-1} is not convexoid. Since S_ρ is a subclass of convexoid operators, we get the required result.

ACKNOWLEDGEMENT

The author is indebted to Dr I. H. Sheth for his valuable guidance, kind help and encouragement in the preparation of this paper.

REFERENCES

Acharya, J. D. (1980). Subclasses of convexoid operators. (Communicated)

Fujii, M. (1971). On some examples of non-normal operators. *Proc. Japan Acad.*, **47**, 458–63.

——— (1973). On some examples of non-normal operators. *Proc. Japan Acad.*, **49**, 118–23.

Furuta, T. (1977). Relations between generalized growth conditions and several classes of convexoid operators. *Can. J. Math.*, **29**, 1010–30.

——— (1978). Convexoid operators and generalized growth conditions associated with unitary ρ-dilations of Sz-Nagy and Foias. *Acta Sci. Math.*, **40**, 53–61.

Holbrook, J. A. R. (1968). On power bounded operators of Sz-Nagy and Foias. *Acta Sci. Math.*, **29**, 299–10.

Luecke, G. R. (1972). Topological properties of paranormal operators. *Trans. Am. math. Soc.*, **172**, 35–43.

Sz-Nagy, and Foias, C. (1970). Harmonic Analysis of Operators on Hilbert Space. North-Holland Publishing Co., Amsterdam.

Patel, S. M., and Gupta, B. C. (1975). Operators satisfying certain growth conditions. *Proc. A.M.S.*, **53**, 341–46.

Patel, S. M. (1976). On some classes of operators associated with operator radii of Holbrook. *Yokohama Math. J.*, **24**, 117–23.