Indian J. pure appl. Math., 11(4) : 462–467, April 1980

STABILITY OF DIFFERENCE AND DIFFERENTIAL EQUATION SYSTEMS

J. N. Kapur, F.N.A. and Q. J. A. Khan

Department of Mathematics, Indian Institute of Technology, Kanpur

(*Received* 24 *August* 1979)

It is shown that the stability of a discrete model represented by a system of difference equations, which is analogous to a continuous model represented by a system of differential equations, depends on the operator used. While in most cases the differential equation system is more stable, a class of operators is found for which the difference equation system is more stable than the corresponding differential equation system.

1. Introduction

The choice between continuous and discrete population models from the point of view of stability has been discussed by a number of authors including Driessche (1974), Duffin (1969), Innes (1974), May (1973a, b) and Usher (1969). In particular May (1973a, b) has considered the continuous model represented by

$$\frac{dN_j}{dt} = F_j\,[N_1(t), N_2(t), \ldots, N_m(t)];\; j = 1, 2, \ldots, m \qquad \ldots(1)$$

and the analogous discrete model

$$N_j(t+1) - N_j(t) = F_j\,[N_1(t), N_2(t), \ldots, N_m(t)];\; j = 1, 2, \ldots, m \qquad \ldots(2)$$

where $N_1(t), N_2(t), \ldots, N_m(t)$ are the populations of the m species and F_j's are non-linear interaction functions. For defining the analogous discrete system, May has replaced the derivative by a forward difference operator with step-size $h > 0$ so that

$$\frac{d}{dt} N_j(t) \approx \frac{1}{h}\,(T-1)\,N_j \qquad \ldots(3)$$

where

$$TN_j(t) = N_j(t+h) \qquad \ldots(4)$$

and time has been normalised so that $h = 1$.

In the two models, the biological features such as trophic structures, birth and death rates, competition and prey-predator interaction are identical. The equilibrium populations are also the same. The main difference is that while for (2), growth takes place is discrete steps, in (1) it is a continuous process. May (1973b) has stated that:

"It is widely understood that difference equations tend to be less stable than their differential equation twins, because the finite time lapse between generations of growth will have the destabilizing effects associated with any time lag in an interacting system. Our discussion makes it explicit; clearly stability of the difference equations system implies stability of the differential equations one, but the converse is not necessarily true".

However Driessche (1974), using the operational rule

$$(T - 1)\, N_j \approx \tfrac{1}{2}h(T + 1)\, dN_j/dt \qquad \text{...(5)}$$

has shown that the resulting analogous discrete system is as stable as the differential equation system.

In the present paper, we examine the operational rule

$$(T - 1)\, N_j \approx \frac{h}{1 + \rho}\,(1 + \rho T)\,\frac{dN_j}{dt} \qquad \text{...(6)}$$

where $\rho(\neq -1)$ is a parameter, to compare the stability of the continuous and discrete models. It is obvious that when $\rho = 1$, (6) reduces to (5) and when $\rho = 0$, (6) reduces to (3) so that the results of Driessche (1974) and May (1973b) should follow as particular cases of our results and we may expect to get additional insight by using other values of ρ.

2. Analytical Discussion

Let $N_j^* \geqslant 0$, $(j = 1, 2, ..., m)$ be the equilibrium population values where

$$F_j(N_1^*, N_2^*, ..., N_m^*) = 0. \qquad \text{...(7)}$$

To discuss the stability of the equilibrium population, we perturb these by small disturbances $v_j(t)$ so that

$$N_j(t) = N_j^*(1 + v_j(t)) \quad (j = 1, 2, ..., m). \qquad \text{...(8)}$$

Substituting in (1), neglecting squares, products and higher powers of $v_j(t)$ and using (7), we get

$$\frac{dv_j}{dt} = \sum_{k=1}^{m} a_{jk} v_k(t); \quad (j = 1, 2, ..., m) \qquad \text{...(9)}$$

where

$$a_{jk} = \frac{N_k^*}{N_j^*}\left(\frac{\partial F_j}{\partial N_k}\right)^* \qquad \text{...(10)}$$

and the partial derivative is evaluated at the equilibrium population values. Substituting

$$v_j(t) = B_j e^{\lambda t} \qquad \text{...(11)}$$

in (9), we find that this can be a solution provided

$$| A - \lambda I | = 0 \qquad \text{...(12)}$$

where A is the $m \times n$ matrix (a_{jk}). The system (1) will be stable if for all eigenvalues of A, we have

$$\text{Re } \lambda < 0. \qquad \text{...(13)}$$

Substituting (8) in (6), we get

$$v_j(t + h) - v_j(t) = \frac{h}{\rho + 1} (1 + \rho T) \frac{dv_j}{dt}. \qquad \text{...(14)}$$

Using (9)

$$v_j(t + h) - v_j(t) = \frac{h}{\rho + 1} \sum_{k=1}^{n} \{a_{jk}[\rho\, v_k(t + h) + v_k(t)]\}. \qquad \text{...(15)}$$

Substituting

$$v_j(t) = C_j \psi^t \qquad \text{...(16)}$$

we find that this will be a solution provided it satisfies the equation

$$\left| A - \frac{\rho + 1}{h} \frac{\psi^h - 1}{\rho\psi^h + 1} I \right| = 0. \qquad \text{...(17)}$$

The analogous discrete model will be stable if

$$| \psi | < 1. \qquad \text{...(18)}$$

3. Geometrical Interpretations

Comparing (12) and (17) we get

$$\lambda = \frac{\rho + 1}{h} \frac{\psi^h - 1}{\rho\psi^h + 1} \qquad \text{...(19)}$$

or

$$\psi^h = \frac{1 + \rho + \lambda h}{1 + \rho - \lambda h}. \qquad \text{...(20)}$$

The circular region $| \psi | < 1$ corresponds to the region

$$| 1 + \rho + \lambda h | < | 1 + \rho - \rho\lambda h |$$

or

$$(1 + \rho + \lambda h)(1 + \rho + \bar{\lambda} h) < (1 + \rho - \rho\lambda h)(1 + \rho - \rho h\bar{\lambda})$$

or

$$\lambda\bar{\lambda}\, h^2(1 - \rho^2) + h(1 + \rho)^2\,(\lambda + \bar{\lambda}) < 0$$

or

$$(x^2 + y^2)\,(1 - \rho^2) + \frac{2x}{h}\,(1 + \rho)^2 < 0. \qquad \ldots(21)$$

(i) If $|\,\rho\,| < 1$, (21) gives

$$\left(x + \frac{1}{h}\,\frac{1 + \rho}{1 - \rho}\right)^2 + y^2 < \frac{1}{h^2}\left(\frac{1 + \rho}{1 - \rho}\right)^2 \qquad \ldots(22)$$

which is the 'interior' of a circle with centre $\left(-\dfrac{1}{h}\,\dfrac{1 + \rho}{1 - \rho},\, 0\right)$, radius $\dfrac{1}{h}\,\dfrac{1 + \rho}{1 - \rho}$ and which passes through the origin.

(ii) If $|\,\rho\,| > 1$, (21) gives

$$\left(x - \frac{1}{h}\,\frac{\rho + 1}{\rho - 1}\right)^2 + y^2 > \frac{1}{h^2}\left(\frac{\rho + 1}{\rho - 1}\right)^2 \qquad \ldots(23)$$

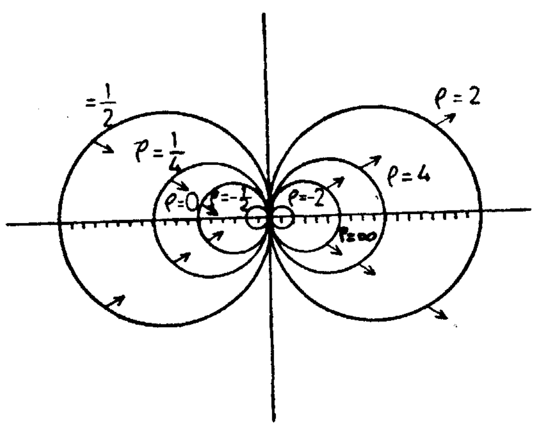
which is the 'exterior' of the circle with centre $\dfrac{1}{h}\,\dfrac{\rho + 1}{\rho - 1}$, radius $\dfrac{1}{h}\,\dfrac{1 + \rho}{\rho - 1}$ and which passes through the origin

(iii) If $\rho = 1$, (21) gives

$$x < 0 \qquad \ldots(24)$$

which is the left-hand half of the λ plane.

We show below, for different values of ρ, the regions corresponding to $|\,\psi\,| < 1$



We get the following possibilities

(i) When $\rho = 0$, we get the interior of the circle of centre $\left(-\frac{1}{h}, 0\right)$ and radius $1/h$. This corresponds to the case discussed by May (1973a, b).

(ii) As ρ increases, we get interiors of larger and larger circles with centres on negative x-axis and all passing through the origin.

(iii) When $\rho = 1$, we get the whole of the left hand plane. This corresponds to the case discussed by Driessche (1974).

(iv) When $-1 < \rho < 0$, we get interiors of circles of radius less than $-1/h$ with centres on negative x-axis and passing through the origin. As $\rho \to -1 + 0$, we get just the point circle at the origin.

(v) When ρ is slightly greater than unity, we get the exterior of a large circle with centre on the positive x-axis and passing through the origin. This exterior includes the entire left-hand half of the x-y plane.

(vi) As ρ increases beyond unity, we get exteriors of smaller and smaller circles with centres on positive x-axis and passing through the origin.

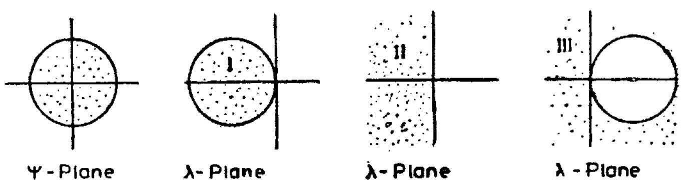
(vii) As $\rho \to \infty$, we get the exterior of a circle of centre $(1/h, 0)$ and radius $1/h$.

(viii) When ρ is negative and large, we get the exterior of a circle of radius larger than $1/h$, centre on the positive x-axis and passing through the origin.

(ix) As $\rho \to -1 -0$, we get the circle of zero radius at the origin.

(x) As $h \to 0$, both the regions given by (22) and (23) approach the region $x < 0$ and in limit both the systems tend to be equistable. This is of course expected.

4. Comparison of the Stability of Continuous and Discrete Models



Case I : $-1 < \rho < 1$

In this case if the difference equation system is stable, $|\psi| < 1$ and the corresponding point in the λ-plane lies within the circle with centre $\left(-\frac{1}{h}\frac{1+\rho}{1-\rho}, 0\right)$ and radius $\frac{1}{h}\frac{1+\rho}{1-\rho}$ i.e. in the shaded region I so that Re $\lambda < 0$ and the differential equation system is also stable.

On the other hand if the differential equation system is stable $\text{Re}\,\lambda < 0$ and the corresponding point lies in the left-hand half of the λ-plane, but this point can be outside shaded region I and $|\psi|$ can be greater than unity so that the difference equations system can be unstable.

Thus in this case the stability of the difference equation system implies the stability of the differential equation system but the converse is not necessarily true.

Case II : $\rho = 1$

In this case $|\psi| < 1 \Rightarrow \text{Re}\,\lambda < 0$ and $\text{Re}\,\lambda < 0 \Rightarrow |\psi| < 1$ so that the two systems are both stable or both unstable. This is the conservative case.

Case III: $|\rho| > 1$

In this case the region $|\psi| < 1$ corresponds to the region III in the λ plane. When $|\psi| < 1$, Re may be positive or negative. Thus $\text{Re}\,\lambda < 0 \Rightarrow |\psi| < 1$, but $|\psi| < 1 \Rightarrow \text{Re}\,\lambda < 0$ so that the stability of the differential equation system implies the stability of the difference equation system, but the converse is not necessarily true.

5. Discussion

We have shown that the relative stability of the difference and differential equations models depends on the operator used and we have demonstrated an operator for which difference equation model is actually more stable.

When $\rho < 1$, difference equation model is less stable, but by making h smaller and smaller, we can make region of stability larger and larger till it coincides with the region of stability for the differential equation.

When $\rho > 1$, differential equation is less stable, but again by making h smaller and smaller, we can make its region of stability larger and larger till it coincides with the region of stability of the difference equation.

When $\rho = 1$, the regions of stability are the same for all small values of h for both the systems.

References

Driessche, P. Van Den (1974). Stability of linear population models. *J. Theor. Biol.*, **48**, 473–76.

Duffin, R. J. (1969). Algorithms for classical stability problems. *Siam Rev.* **11**, 196–213.

Innes, G. (1974). In defence of difference equations in "Lecture Notes in Biomathematics", Vol. 2 (ed. S. Levin). Spring-Verlag, New York, pp. 102–22.

May, R. M. (1973a). On relationships among various types of population models. *Am. Nat.*, **101**, 46–57.

——————— (1973b). Stability and Complexity in Model Ecosystem, Princeton University Press, New Jersey.

Usher, M. B. (1969). Developments in Leslie Matrix Model in Mathematical Models in Ecology (J.N.R. Jeffers). Blackwell Scientific Publications, Oxford, pp. 29–61.