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ON ORTHOGONAL POLYNOMIALS OF TWO VARIABLES

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Polynomial solutions of two differential equations of Krall and Sheffer (1967) related to orthogonal polynomials are obtained. Recurrence relations and generating functions of these polynomial solutions are also discussed.

§1. Krall and Sheffer (1967) characterized the orthogonal polynomials in two variables through a set of differential equations. Among the various differential equations they encountered all of them were solved except four of them, viz. eqns. (1.1), (1.2), (1.3) and (1.4)

$$3yW_{xx} + 2W_{xy} - (xW_x + yW_y) + nW = 0 \quad ...(1.1)$$

$$(x + \alpha)\, W_{xx} + 2(y + 1)\, W_{yy} + (xW_x + yW_y) - nW = 0 \quad ...(1.2)$$

$$x^2 W_{xx} + 2xyW_{xy} + (y^2 - y)\, W_{yy} + g(x - 1)\, W_x + g(y - \alpha)\, W_y - n(n - 1 + g)\, W = 0 \quad ...(1.3)$$

$$(x^2 + y + 1)\, W_{xx} + (2xy + 2x)\, W_{xy} + (y^2 + 2y + 1)\, W_{yy} + g(xW_x + yW_y) - n(n - 1 + g)\, W = 0. \quad ...(1.4)$$

The solutions of above differential equations are of the type

$$P_{n-k,k}(x, y) = x^{n-k}y^k + \text{lower degree terms.}$$

$$(k = 0, 1, 2, ..., n \text{ and } n = 0, 1, 2, ...).$$

The solution of (1.1) was given in the closed form by Anthony Du Rapau (1967) as

$$T_{n-k,k}(x, y) = \sum_{q=0}^{n-k} \sum_{p=\max(0,\, q-k)}^{[\frac{1}{2}(n-k-q)]} A_{n-k-2p-q,k+p-q}\, x^{n-k-2p-q}\, y^{k+p-q} \quad ...(1.5)$$

where
$$A_{n-k-2p-q,k+p-q} = \frac{(-)^p\, (3)^{p-q}\, 2^{q-k}}{(p - q + k)!\, q!} (-n + k)_{2p+q} \sum_{r=0}^{k} (-)^k\, 2^r \binom{k}{r} \times (-q)_{k-r}\, (-p - r)_r.$$

Whereas the solution of (1.2) was obtained later on by Krall and Sheffer in the form (unpublished research notes)

$$M_{ij}(x, y) = \sum_{k=0}^{i} \sum_{l=0} \frac{2^l(-i)_k\,(-i+\alpha+1)_k\,(-j)_l\,(-j+3/2)_l}{k!\;l!}$$

$$\times\;(x+\alpha)^{i-k}\,(y+1)^{j-l}. \quad \ldots(1.6)$$

It might be of interest to note that this solution can be expressed as the product of two Laguerre polynomials, viz.,

$$M_{ij}(x, y) = 2^j i!\; j!\; L_i^{(-\alpha-1)}\,(-x-\alpha)\;\; L_j^{(-3/2)}\left(-\frac{y+1}{2}\right) \quad \ldots(1.7)$$

where Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\;{}_1F_1(-n;\,1+\alpha;\,x).$$

The purpose of this paper is to investigate the solutions of the remaining two differential eqns. (1.3) and (1.4). These solutions, as indicated by Krall and Sheffer (1967), are non-classical orthogonal polynomials. The recurrence relations and generating functions of the polynomial solutions of (1.3) and (1.4) are also obtained.

§2. Assuming the solution of (1.3) in the form

$$W = R_{n-k,k}(x, y) = \sum_{r=0}^{\infty}\;\sum_{s=0}^{\infty} A_{r,s}x^ry^s \quad \ldots(2.1)$$

where $A_{r,s} = 0$ when $r+s>n$ and $A_{n-j,k} = \begin{bmatrix} 0 \text{ if } j \neq k \\ 1 \text{ if } j = k \end{bmatrix}$

and substituting (2.1) in the differential equation (1.3) and equating the coefficients of like powers of x^ry^s, we have

$$[(r+s)^2-(r+s)+g(r+s-n)-n^2+n]\,A_{r,s}$$

$$= (s+1)\,(s+g\alpha)\,A_{r,s+1} + g(r+1)\,A_{r+1,s}. \quad \ldots(2.2)$$

The solution of this recurrence relation is

$$A_{n-k,k-3} = \frac{(-k)_3\,(-k-g\alpha+1)_3}{3!\,(2-2n-g)_3}$$

$$A_{n-k-1,k-2} = \frac{3g(n-k)\,(-k-g\alpha+1)_2\,(-k)_2}{3!\,(2-2n-g)_3}$$

$$A_{n-k-2,k-1} = \frac{3g^2(-n+k)_2\,k(k+g\alpha-1)}{3!\,(2-2n-g)_3}$$

$$A_{n-k-3,k} = \frac{(-g)^3\,(-n+k)_3}{3!\,(2-2n-g)_3}.$$

The symmetry of the above values of $A_{r,s}$ gives us the solution of (1.3) viz.

$$R_{n-k,k}(x, y, g, \alpha) = \frac{(-)^n g^{n-k}(g\alpha)_k}{(g+n-1)_n} \times \sum_{j=0}^{n-k} \sum_{p=0}^{k} \frac{(-n+k)_j (-k)_p (g+n-1)_{j+p}}{j!\, p!\, (g\alpha)_p} \left(\frac{x}{g}\right)^j y^p \quad \ldots(2.3)$$

(where $k = 0, 1, 2, \ldots, n$ and $n = 0, 1, 2, \ldots$)

which could be verified directly by substituting (2.3) in the differential equation. (1.3).

For the sake of simplicity alone we consider the polynomial $R_{n-k,k}(gx, y, g, (\alpha/g))$ and shall denote it by $P_{n-k,k}(x, y, g, \alpha)$ i.e.

$$P_{n-k,k}(x, y, g, \alpha) = \frac{(-)^n g^{n-k}(\alpha)_k}{(g+n-1)_n} \times \sum_{j=0}^{n-k} \sum_{p=0}^{k} \frac{(-n+k)_j (-k)_p}{j!\, p!\, (\alpha)_p} (g+n-1)_{j+p}\, x^j y^p. \quad \ldots(2.4)$$

To economize further we would only exhibit those of the parameters g and α which undergo change in a particular equation and if none of x, y, g and α undergoes change in a particular equation the polynomial would be simply denoted by $P_{n-k,k}$.

It may be worth mentioning that $P_{n-k,k}$ is nothing else but the sum of $n-k+1$ index dependent Jacobi polynomials $P_k^{(\alpha-1, g+n+j-1-\alpha-k)}(1-2y)$. In fact, we have

$$P_{n-k,k} = \frac{(-)^n g^{n-k}\, k!}{(g+n-1)_n} \sum_{j=0}^{n-k} \frac{(-n+k)_j (g+n-1)_j}{j!} x^j \times P_k^{(\alpha-1, g+n+j-1-\alpha-k)}(1-2y). \quad \ldots(2.5)$$

Making use of the integral representation for the Jacobi polynomial (Rainville 1960) we obtain the following integral representation for the polynomial $P_{i,n-i}$:

$$P_{i,n-i} = \frac{(-)^n g^i \Gamma(\alpha+n-i)}{\Gamma(g+2n-1)\,\Gamma(\alpha-g-n+1)} \int_0^1 t^{g+n-2}(1-t)^{\alpha-g-n} \times (1-yt)^{n-i}\, Y_i\left(g+n-\alpha-i; \frac{-xt}{1-t}\right) dt \quad \ldots(2.6)$$

where $Y_m(c; x)$, the Bessel polynomial, is defined as

$$Y_m(c; x) = {}_2F_0(-m, c+m; -; x).$$

We could have also put $P_{i,n-i}$ as the sum of $n - i + 1$ index dependent Bessel polynomials $Y_i(g + n + p - i - 1; x)$:

$$P_{i,n-i} = \frac{(-)^n g^i(\alpha)_{n-i}}{(g+n-1)_n} \sum_{p=0}^{n-i} \frac{(-n+i)_p (g+n-1)_p y^p}{p! (\alpha)_p} \times Y_i(g+n+p-i-1; x). \quad \ldots(2.7)$$

Using the integral representation for the Bessel polynomial (Al-Salam 1957) we get the integral representation of the polynomial $P_{n-k,k}$:

$$P_{n-k,k} = \frac{(-)^n g^{n-k} k!}{\Gamma(g+2n-1)} \int_0^\infty e^{-t} t^{g+n-2} (1 - xt)^{n-k} L_k^{(\alpha-1)}(yt)\, dt. \quad \ldots(2.8)$$

It is of interest to note that the polynomials $P_k^{(\alpha-1, g+n+j-1-\alpha-k)}(1-2y)$ and $Y_i(g + n + p - i - 1; x)$ are non-orthogonal polynomials in one variable whereas their finite linear combinations as given by (2.5) and (2.7) yield orthogonal polynomials of two variables. Next we prove a relation expressing $P_{i,n-i}$ as a sum of $i + 1$ products of orthogonal Jacobi and Bessel polynomials, viz.,

$$P_{i,n-i} = \frac{(-)^n g^i(\alpha)_{n-i} (n-i)!}{(g+n-1)_n} \times \sum_{r=0} \frac{(-)^r (-i)_r (g+n-1)_r (2n-2i+g-1)_r (2n-2i+g-1+2r)}{r! (\alpha)_{n-2i+r} (2n-2i+g-1)_{i+r+1}} \times Y_r(2n-2i+g-1; x)\, P_{n-2i+r}^{(\alpha-1, 2i+g-\alpha-1)}(1-2y). \quad \ldots(2.9)$$

In order to obtain (2.9) we start by replacing x^j in the Definition (2.4) of the polynomial $P_{i,n-i}$ by using the relation (Rainville 1960, p. 294)

$$x^j = \sum_{r=0}^{j} \binom{j}{r} \frac{(-)^r (c)_r (c+2r)}{(c)_{j+r+1}} Y_r(c; x) \text{ for } c = 2n - 2i + g - 1,$$

to obtain

$$P_{i,n-i} = \frac{(-)^n g^i(\alpha)_{n-i}}{(g+n-1)_n} \sum_{p=0}^{n-i} \frac{(-n+i)_p (g+n-1)_p y^p}{p! (\alpha)_p} \times$$

(*equation continued on p.* 496)

$$\times \sum_{r=0} \frac{(-i)_r (g+n-1+p)_r (2n-2i+g-1)_r (2n-2i+g-1+2r)(-)^r}{r!\,(2n-2i+g-1)_{2r+1}}$$

$$\times\, {}_2F_1\left[\begin{matrix}-i+r,\ g+n-1+r+p;\\ 2n-2i+g+2r\end{matrix}\right] Y_r(2n-2i+g-1;\,x). \qquad (2.10)$$

To obtain the last line we use the Gauss' summation theorem for ${}_2F_1(1)$ (Rainville 1960). Using the definitions of Jacobi and Bessel polynomials we get (2.9).

We may deduce from (2.9) that the set of polynomials $\{P_{i,n-i}(Z, y)\}$ are orthogonal with respect to the weight function

$$\rho(Z, y) = 2^{2i+g-1} y^{\alpha-1}(1-y)^{2i+g-\alpha-1}\, \frac{1}{2\pi i} \sum_{l=0}^{\infty} \frac{(2n-2i+g-1)}{(2n-2i+g-1)_l}\, Z^{-l} \qquad \text{...(2.11)}$$

where the integration with respect to Z is carried out on the unit circle $|Z| = 1$ and with respect to y from 0 to 1 respectively.

The moments α_{mn} associated with the polynomial (2.4) are

$$\alpha_{mn} = \frac{2^{2i+g-1}}{(2n-2i+g)_n}\, \frac{\Gamma(\alpha+m)\,\Gamma(2i+g-\alpha)}{\Gamma(2i+m+g)} \qquad \text{...(2.12)}$$

and the norm of $P_{i,n-i}$, is

$$\int\int_R [P_{i,n-i}(Z, y, g, \alpha)]^2\, \rho(Z, y)\, dZ\, dy$$

$$= \frac{2^{2i+g-1} g^{2i} (2n-2i+g-1)\,(n-i)!)^2\,(\alpha)_{n-i}}{(n-2i)!\,(\alpha)_{n-2i}\,(g+n-1)_n}$$

$$\frac{\Gamma(\alpha+n-i)\,\Gamma(n+g-\alpha)}{[(2n-2i+g-1)_{i+1}]^2\,\Gamma(g+2n-1)}$$

$$\times\, {}_5F_4\left[\begin{matrix}2n-2i+g-1,\ n+g-\alpha,\ n+g-1,\ -i,\ -i;\ -1\\ n-2i+\alpha,\ n-2i+1,\ 2n-i+g,\ 2n-i+g\end{matrix}\right]. \qquad \text{...(2.13)}$$

Straightforward manipulations show that the polynomials $\{P_{i,n-i}\}$ satisfy the recurrence formulae

$$(2n+g-1)_2\,[x(P_{i+1,n-i})_x - (i+1)\,P_{i+1,n-i}] - g(i+1)\,[x(P_{i,n-1})_x$$

$$+ (2n-i+g-1)\,P_{i,n-i}] - (n-i)(n-i+\alpha-1)\,[x(P_{i+1,n-i-1})_x$$

$$- (i+1)\,P_{i+1,n-i-1}] = 0 \qquad \text{...(2.14)}$$

and

$$(2n + g - 1)_2 \, [y(P_{i,n-i+1})_y - (n - i + 1) \, P_{i,n-i+1}] - (n - i + 1) \times (n - i + \alpha) \, [y(P_{i,n-i})_y + (n + i + g - 1) \, P_{i,n-i}] - ig \, [y(P_{i-1,n-i+1})_y - (n - i + 1) \, P_{i-1,n-i+1}] = 0. \quad \ldots(2.15)$$

The differential recurrence relations with a change in the parameter g are

$$(g + 2)^{i-1} \, [P_{i,n-i}(g)]_x = ig^i P_{i-1,n-i}(g + 2) \quad \ldots(2.16)$$

$$(g + 1)^{i-1} \, (2n + g - 2) \, x \, [P_{i,n-i}(g)]_x = i(2n + g - 2)(g + 1)^{i-1} \, P_{i,n-i}(g) + ig^i P_{i-1,n-i}(g + 1) \quad \ldots(2.17)$$

$$(2n + g - 2)(g + 1)^i \, y[P_{i,n-i}(g)]_y = (n - i)(2n + g - 2)(g + 1)^i \, P_{i,n-i}(g) + (n - i)(n - i + \alpha - 1) \, g^i P_{i,n-i-1}(g + 1). \quad \ldots(2.18)$$

Using (2.16) and (2.17) we can get the following pure recurrence relation satisfied by $P_{i,n-i}$:

$$(g - 2)^i \, (g - 1)^{i-1} \, (2n + g - 4) \, x P_{i-1,n-i}(g) = g^{i-1}(g - 2)^i \, P_{i-1,n-i}(g - 1) + g^{i-1}(g - 1)^{i-1} \, (2n + g - 4) \, P_{i,n-i}(g - 2). \quad \ldots(2.19)$$

Next, we observe that

$$\triangle_\alpha P_{n-k,k}(g, \alpha) = \frac{kg^{n-k}}{(g + 1)^{n-k} \, (2 - g - 2n)} \, P_{n-k,k-1}(g + 1, \alpha + 1) \quad \ldots(2.20)$$

(where $\triangle_\alpha f(\alpha, \beta) = f(\alpha + 1, \beta) - f(\alpha, \beta)$

and so in general,

$$\triangle_\alpha^s \, P_{n-k,k}(g, \alpha) = \frac{(-)^s \, (-k)_s \, g^{n-k}}{(2 - g - 2n)_s \, (g + s)^{n-k}} \, P_{n-k,k-s}(g + s, \alpha + s). \quad \ldots(2.21)$$

Making use of Newton's formula

$$f(\alpha + m) = \sum_{s=0}^{m} \binom{m}{s} \triangle_\alpha^s \, f(\alpha) \quad \ldots(2.22)$$

and (2.21) we get the following general recurrence formula satisfied by $P_{n-k,k}(\alpha + m)$:

$$P_{n-k,k}(g, \alpha + m) = \sum_{s=0}^{m} \binom{m}{s} \frac{(-)^s \, (-k)_s \, g^{n-k}}{(2 - g - 2n)_s \, (g + s)^{n-k}} \times P_{n-k,k-s}(g + s, \alpha + s). \quad \ldots(2.23)$$

Similarly we may obtain

$$P_{i,n-i}(g+m,\alpha) = \frac{(g-1)_{2n}\,(g+m)^i}{(g+m+n-1)_n} \sum_{r=0}^{m} \sum_{s=0}^{m-r} \binom{m}{s+r}$$
$$\times \binom{s+r}{r} \frac{(-i)_s\,(-n+i)_r}{(g-1)_{n+s+r}} \cdot (g+2s+2r)^{s-i}$$
$$\times (-x)^s\,(-y)^r\, P_{i-s,n-i-r}\,(g+2s+2r,\alpha+r). \qquad \ldots(2.24)$$

We also notice that the polynomials $\{P_{i,n-i}\}$ have a generating function

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(g-1)_n}{(g-n)^i\,(\alpha)_{n-i}} \frac{s^i t^{n-i}}{i!(n-i)!} P_{i,n-i}(g-n)$$
$$= (1-xs)^{1-g}\, e^{-s-t}\, {}_1F_1\left[g-1;\alpha;\frac{yt}{1-xs}\right]. \qquad \ldots(2.25)$$

For proving (2.25) we replace g by $(g-n)$ in (2.4) and then on multiplying both the sides by $\dfrac{(g-1)_n\, s^i t^{n-i}}{(\alpha)_{n-i}\,(g-n)^i\, i!\,(n-i)!}$ and summing for i from 0 to n and then n from 0 to ∞ we have

$$S \equiv \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(g-1)_n\, s^i t^{n-i}}{(g-n)^i\,(\alpha)_{n-i}\, i!\,(n-i)!} P_{i,n-i}(g-n)$$
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-)^n\, s^i t^{n-i}}{i!\,(n-i)!} \sum_{j=0}^{i} \sum_{p=0}^{n-i} \frac{(-i)_j\,(-n+i)_p}{j!\,p!\,(\alpha)_p}$$
$$\times (g-1)_{j+p}\, x^j y^p. \qquad \ldots(2.26)$$

In the R.H.S. of (2.26) rearranging the series we get

$$S = \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{(g-1)_{j+p}\,(xs)^j\,(yt)^p}{p!\,(\alpha)_p\, j!} \frac{(-s)^i}{i!} \frac{(-t)^n}{n!}$$
$$= (1-xs)^{1-g}\, e^{-s-t}\, {}_1F_1\left[g-1;\alpha;\frac{yt}{1-xs}\right].$$

§3. We begin this section by presenting a solution of the differential equation (1.4) in a closed form. On replacing $y+1$ by y (1.4) reduces to

$$(x^2+y)\, W_{xx} + 2xyW_{xy} + y^2 W_{yy} + gxW_x + g(y-1)\, W_y$$
$$- n(n-1+g)\, W = 0. \qquad \ldots(3.1)$$

Next, assuming the solution of (3.1) in the form (2.1) and proceeding on the same lines as in §2, we find that $A_{r,s}$ satisfy the recurrence relation

$$[(r+s)^2-(r+s)+g(r+s-n)-n^2+n]\,A_{r,s} = g(s+1)\,A_{r,s+1}-(r+1)_2\,A_{r+2,s-1}. \quad \ldots(3.2)$$

Solving this recurrence relation, we get the solution of (3.1) viz:

$$Q_{n-k,k}(x,y,g)=\sum_{j=0}^{[(n-k)/2]}\sum_{p=0}^{k+j}\frac{(-)^{j+p}\,g^p(-n+k)_{2j}}{j!\,p!\,(2-2n-g)_{p+j}} \times\sum_{l=0}^{\min(j,p)}\frac{(-j)_l\,(-p)_l\,(-k)_{p-l}}{2^l\,l!}(-)^l\,x^{n-k-2j}y^{k-p+j} \quad \ldots(3.3)$$

which may be verified by a direct substitution of (3.3) in (3.1). Here $[l/2]$ means the greatest integer $\leqslant [l/2]$. For the sake of convenience we would abbreviate $Q_{n-k,k}(x,y,g)$ simply as $Q_{n-k,k}$ and $Q_{n-k,k}(g^{1/2}x,gy,g)$ as $Q_{n-k,k}(g)$.

It can be shown that $Q_{n-k,k}(g)$ satisfy

$$g^{n-i}\,[Q_{2i+1,n-2i}(g-2)]_x=(2i+1)\,(g-2)^{(2n-2i+1)/2}\,Q_{2i,n-2i}(g) \quad \ldots(3.4)$$

$$g^{(2n-2i-1)/2}\,[Q_{2i,n-2i}(g-2)]_x=2i(g-2)^{n-i}\,Q_{2i-1,n-2i}(g) \quad \ldots(3.5)$$

$$\begin{aligned}(g^2+g)^{(2n-2i-1)/2}\,(2n+g-2)\,[Q_{2i+1,n-2i}(g-2)]_y &= (g-2)^{(2n-2i+1)/2}\,(g+1)^{(2n-2i-1)/2}\\ &\quad\times(n-2i)\,(2n+g-2)\,Q_{2i+1,n-2i-1}(g)+2i(2i+1)\,g^{(2n-2i-1)/2}\\ &\quad\times(g-2)^{(2n-2i+1)/2}\,Q_{2i-1,n-2i}(g+1)\end{aligned} \quad \ldots(3.6)$$

$$\begin{aligned}(2n+g-3)\,g^{(2n-2i-1)/2}(g-1)^{i-n}\,[Q_{2i,n-2i}(g-1)]_x &= 2i\,\{y[Q_{2i-1,n-2i}(g)]_y+x[Q_{2i-1,n-2i}(g)]_x+(g+n-2)\,Q_{2i-1,n-2i}(g)\}\end{aligned} \quad \ldots(3.7)$$

$$\begin{aligned}g^{n-i}(g-1)^{(2n-2i+1)/2}\,y\,[Q_{2i+1,n-2i}(g-2)]_y &= g^{n-i}(g-2)^{(2n-2i+1)/2}\,(2n+g-1)\,Q_{2i+1,n-2i}(g-1)\\ &\quad-g^{n-i}(g-1)^{(2n-2i+1)/2}\,(n+g-2)\,Q_{2i+1,n-2i}(g-2)\\ &\quad-(2i+1)(g-1)^{(2n-2i+1)/2}\,(g-2)^{(2n-2i+1)/2}x\,Q_{2i,n-2i}(g)\end{aligned} \quad \ldots(3.8)$$

$$\begin{aligned}(g^2+g)^{n-i-1}\,(2n+g-4)\,[Q_{2i,n-2i}(g-2)]_y &= (g+1)^{n-i-1}(g-2)^{n-i}\,(n-2i)\,(2n+g-4)\,Q_{2i,n-2i-1}(g)\\ &\quad+2i(2i-1)\,g^{n-i-1}(g-2)^{n-i}\,Q_{2i-2,n-2i}(g+1)\end{aligned} \quad \ldots(3.9)$$

$$(2i+1)\, y\, [Q_{2i,n-2i}(g)]_y = (g+2n-1)\, g^{n-i}(g-1)^{(2i-2n-1)/2} \times [Q_{2i+1,n-2i}(g-1)]_x - (2i+1)\, \{x\, [Q_{2i,n-2i}(g)]_x + (g+n-1)\, Q_{2i,n-2i}(g)\} \qquad ...(3.10)$$

and

$$g^{(2n-2i-1)/2}(g-1)^{n-i}\, y\, [Q_{2i,n-2i}(g-2)]_y = g^{(2n-2i-1)/2}(g-2)^{n-i}\,(2n+g-3)\, Q_{2i,n-2i}(g-1) - g^{(2n-2i-1)/2} \times (g-1)^{n-i}\,(n+g-3)\, Q_{2i,n-2i}(g-2) - 2i(g-1)^{n-i} \times (g-2)^{n-i}\, xQ_{2i-1,n-2i}(g). \qquad ...(3.11)$$

(3.7), on using (3.4) – (3.6), yields the following pure recurrence relation

$$(2n+g-3)\,(g+1)^{(2i-2n+1)/2}\, Q_{2i-1,n-2i}(g+1) = y(n-2i)\,(g+2)^{(2i-2n+3)/2}\, Q_{2i-1,n-2i-1}(g+2) + y(2i-2)_2\, (2n+g-4)^{-1}\,(g+3)^{(2i-2n+3)/2}\, Q_{2i-3,n-2i}(g+3) + x(2i-1)\,(g+2)^{i-n+1}\, Q_{2i-2,n-2i}(g+2) + (g+n-2) \times g^{(2i-2n+1)/2} Q_{2i-1,n-2i}(g). \qquad ...(3.12)$$

Similarly, (3.10) in view of (3.4), (3.5), (3.9), gives the pure recurrence relation

$$y(2i+1)\,(n-2i)\,(g+2)^{i-n+1}\, Q_{2i,n-2i-1}(g+2) + y(2i-1)_3 \times (2n+g-2)^{-1}\,(g+3)^{i-n+1}\, Q_{2i-2,n-2i}(g+3) = (2i+1)\,(g+2n-1)\,(g+1)^{i-n}\, Q_{2i,n-2i}(g+1) - xQ_{2i-1,n-2i}(g+2)\,(2i)_2\,(g+2)^{(2i-2n+1)/2} - (2i+1)\,(g+n-1)\, g^{i-n} Q_{2i,n-2i}(g). \qquad ...(3.13)$$

Using (3.4) – (3.11) we may obtain other pure recurrence relations. Next, we prove that according as i is even or odd generating functions of $Q_{i,n-i}(g)$ are

$$\sum_{n=0}^{\infty} \sum_{i=0}^{[n/2]} \frac{(g-1)_n\, s^i t^{n-i} Q_{2i,n-2i}(g-n)}{(n-2i)!\,(2i)!\,(g-n)^{n-i}} = (1-yt-yst)^{1-g}\, e^{-t(s+2)/2} \times {}_2F_1\left[\frac{g-1}{2}, \frac{g}{2}; \tfrac{1}{2}; \frac{x^2 st}{(1-yt-yst)^2}\right] \qquad ...(3.14)$$

and

$$\sum_{n=0}^{\infty} \sum_{i=0}^{[n/2]} \frac{(g+1)_n s^i t^{n-i} Q_{2i+1,n-2i}(g-n)}{(n-2i)!\,(2i+1)!\,(g-n)^{(2n-2i+1)/2}} = \frac{xe^{-t(s+2)/2}}{(1-yt-yst)^{1+g}} \times {}_2F_1\left[\frac{g+1}{2}, \frac{g}{2}+1; \tfrac{3}{2}; \frac{x^2 st}{(1-yt-yst)^2}\right]. \qquad ...(3.15)$$

For proving (3.14) we replace g by $(g - n)$ in the definition of $Q_{2i,n-2i}(g)$ and multiply both sides by $\frac{(g-1)_n s^i t^{n-i}}{(n-2i)!\,(2i)!\,(g-n)^{n-i}}$, sum for i from 0 to $\left[\frac{n}{2}\right]$ and then for n from 0 to ∞, we have

$$S \equiv \sum_{n=0}^{\infty} \sum_{i=0}^{[n/2]} \frac{(g-1)_n s^i t^{n-i} Q_{2i,n-2i}(g-n)}{(n-2i)!\,(2i)!\,(g-n)^{n-i}}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{[n/2]} \frac{(-)^{n-i} s^i t^{n-i}}{i!\,(n-i)!} \sum_{j=0}^{i} \sum_{p=0}^{n-i-j}$$

$$\times \sum_{l=0}^{\min(i-j,\,n-i-p-j)} \frac{(-)^j (-i)_{j+l} (-n+i)_{j+p+l} (g-1)_{2j+p} x^{2j} y^p}{j!\,l!\,(\frac{1}{2})_j (j+p+i-i)!\,4^j 2^l}.$$

...(3.16)

In the R.H.S. of (3.16) rearranging the series we obtain

$$S = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-)^{p+i} (g-1)_{2j+n+i}}{j!\,(\frac{1}{2})_j\, n!\, i!\, p!\, l!} \left(\frac{x^2 st}{4}\right)^j$$

$$\times (yt)^n (yts)^i t^p \left(\frac{st}{2}\right)^l \qquad ...(3.17)$$

summing the n, i, p and l series we get (3.14). The proof of (3.15) follows on similar lines.

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