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MAPPINGS ON METRIC SPACES

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It is well known that a contraction on a complete metric space has a unique fixed point. Kannan (1968) exhibited another class of maps with this property and investigated about common fixed points of pairs of mappings. Fisher (1976) defined Kannan maps and obtained a relationship between contractions and Kannan maps. Our first result exhibits a wide class of mappings each member of which has a unique fixed point such that contractions and Kannan maps belong to it. We call the members of this class as generalized Kannan maps. Our second result exhibits a relationship between generalized Kannan maps satisfying a condition and contractions.

§1. A contraction mapping is a mapping T on a metric space (X, d) into itself which satisfies $d(Tx, Ty) \leqslant Cd(x, y)$ for all x, y in X where $0 \leqslant C < 1$. It is well known that a contraction mapping on a complete metric space has a unique fixed point. Kannan (1968) investigated the conditions under which two mappings on a metric space have a common fixed point and proved the following:

Theorem (Kannan 1968) — If T_1 and T_2 are mappings on a complete metric space (X, d) into itself and if there is a constant K such that $0 \leqslant K < \frac{1}{2}$ and

$$d(T_1x, T_2y) \leqslant K(d(x, T_1x) + d(y, T_2y)) \qquad \text{...(A)}$$

for all x, y in X, then T_1 and T_2 have a unique common fixed point.

Fisher (1976) defined a mapping T on a metric space (X, d) to be a Kannan mapping if it satisfies (A) with $T_1 = T_2 = T$ and obtained a relation between Kannan mappings and contraction mappings. In fact he proved that if T is a contraction mapping on a metric space into itself then T^n is a Kannan mapping for large n and on the other hand if T is any Kannan mapping on (X, d) satisfying

$$d(x, Tx) + d(y, Ty) \leqslant hd(x, y)$$

for all x, y in X and a fixed $h > 0$, T^n is a contraction mapping for large n.

The purpose of this note is to extend these results to a much wider class of mappings.

§2. Let (X, d) be a metric space.

Definition 1 — A pair of mappings (T_1, T_2) on X into itself is said to have the Kannan property or simply the property K if there exist constants $K_i (1 \leqslant i \leqslant 5)$ so that

$$0 \leqslant K_i \text{ for } 1 \leqslant i \leqslant 5 \quad \ldots(1.1)$$

$$K_1 + K_2 + 2K_i + K_5 < 1 \text{ for } i = 3 \text{ and } i = 4 \quad \ldots(1.2)$$

and

$$d(T_1 x, T_2 y) \leqslant K_1 d(x, T_1 x) + K_2 d(y, T_2 y) + K_3 d(x, T_2 y) + K_4 d(y, T_1 x) + K_5 d(x, y) \quad \ldots(1.3)$$

for all x, y in X.

Remark : The inequalities (1.2) imply that $K_3 + K_4 + K_5 < 1$.

Definition 2 — A mapping T on X into itself is said to be a generalized Kannan mapping if the pair (T, T) has the property K. In that case we sometimes write that T has the property K.

Remark : Contraction mappings and Kannan mappings are generalized Kannan mappings.

Theorem 1 — If T is a generalized Kannan mapping of a complete metric space into itself, then T has a unique fixed point.

Theorem 1 is an immediate consequence of the following proposition.

Proposition 1 — Let (X, d) be a complete metric space and T_1, T_2 be mappings of X into itself. If (T_1, T_2) has the property K then (T_1, T_2) has a unique common fixed point.

PROOF : Let $x \in X$ and write $x_1 = T_1 x$, $x_2 = T_2 x_1$, $x_3 = T_1 x_2$, and so on. Then

$$d(x_1, x_2) = d(T_1 x, T_2 x_1) \leqslant K_1 d(x, T_1 x) + K_2 d(x_1, T_2 x_1) + K_3 d(x, T_2 x_1) + K_4 d(x_1, T_1 x) + K_5 d(x, x_1) \text{ where}$$

$K_i (1 \leqslant i \leqslant 5)$ are constants as in Definition 1.

Hence $(1 - K_2)\, d(x_1, x_2) \leqslant (K_1 + K_5)\, d(x, x_1) + K_3 d(x, x_2)$

$$\leqslant (K_1 + K_5)\, d(x, x_1) + K_3(d(x, x_1) + d(x_1, x_2)).$$

$$\Rightarrow (1 - K_2 - K_3)\, d(x_1, x_2) \leqslant (K_1 + K_3 + K_5)\, d(x, x_1).$$

Hence $d(x_1, x_2) \leqslant \dfrac{K_1 + K_3 + K_5}{1 - K_2 - K_3}\, d(x, x_1)$.

An inductive argument yields the inequality

$$d(x_n, x_{n+1}) \leqslant r^n d(x, x_1)$$

where $$r = \frac{K_1 + K_3 + K_5}{1 - K_2 - K_3}.$$

Now $$\begin{aligned} d(x_n, x_{n+p}) &\leqslant d(x_n, x_{n+1}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leqslant (r^n + r^{n+1}, \ldots + r^{n+p-1})\, d(x, x_1) \\ &< \frac{r^n}{1-r}\, d(x, x_1). \qquad \ldots\text{(B)} \end{aligned}$$

Since $0 \leqslant r < 1$, it is now clear that $\{x_n\}$ is a Cauchy sequence and since X is complete, there is an x_0 in X so that $d(x_n, x_0) \to 0$ as $n \to \infty$.

We show that x_0 is a common fixed point of T_1 and T_2. Let n be any even integer.

$$\begin{aligned} d(x_0, T_1x_0) \leqslant d(x_0, x_n) + d(x_n, T_1x_0) \leqslant d(x_0, x_n) &+ K_1 d(x_0, T_1x_0) \\ &+ K_2 d(x_{n-1}, T_2x_{n-1}) + K_3 d(x_0, T_2x_{n-1}) \\ &+ K_4 d(T_1x_0, x_{n-1}) + K_5 d(x_{n-1}, x_0). \end{aligned}$$

Hence $$\begin{aligned} (1 - K_1)\, d(x_0, T_1x_0) - K_4 d(T_1x_0, x_{n-1}) \leqslant d(x_0, x_n) &+ K_2 d(x_{n-1}, x_n) \\ &+ K_3 d(x_0, x_n) + K_5 d(x_{n-1}, x_0). \end{aligned}$$

Let $n = 2p$. Taking the limits as $p \to \infty$, we get

$$(1 - K_1)\, d(x_0, T_1x_0) - K_4 d(T_1x_0, x_0) \leqslant 0.$$

Since $K_1 + K_4 < 1$, we now get $d(x_0, T_1x_0) = 0$. Hence $x_0 = T_1x_0$. Now let n be any odd integer.

$$\begin{aligned} d(x_0, T_2x_0) &\leqslant d(x_0, x_n) + d(x_n, T_2x_0) = d(x_0, x_n) + d(T_1x_{n-1}, T_2x_0) \\ &\leqslant d(x_0, x_n) + K_1 d(x_{n-1}, T_1x_{n-1}) + K_2 d(x_0, T_2x_0) \\ &\qquad + K_3 d(x_{n-1}, T_2x_0) \\ &\qquad + K_4 d(x_0, T_1x_{n-1}) + K_5 d(x_{n-1}, x_0). \end{aligned}$$

Hence $$\begin{aligned} &(1 - K_2)\, d(x_0, T_2x_0) - K_3 d(x_{n-1}, T_2x_0) \\ &\qquad \leqslant d(x_0, x_n) + K_1 d(x_{n-1}, x_n) + K_4 d(x_0, x_n) + K_5 d(x_{n-1}, x_0). \end{aligned}$$

Let $n = 2p - 1$. Taking the limits as $p \to \infty$ we get

$$(1 - K_2 - K_3)\, d(x_0, T_2x_0) \leqslant 0,$$

and since $K_2 + K_3 < 1$ we get $x_0 = T_2x_0$. Thus x_0 is a common fixed point of T_1 and T_2.

We now prove the uniqueness. Suppose y_0 is also a common fixed point of T_1 and T_2. Then $d(x_0, y_0) = d(T_1x_0, T_2y_0)$. Hence

$$d(x_0, y_0) \leqslant K_1d(x_0, T_1x_0) + K_2d(y_0, T_2y_0) + K_3d(x_0, T_2y_0) + K_4d(y_0, T_1x_0) + K_5d(x_0, y_0).$$

Hence $d(x_0, y_0) \leqslant (K_3 + K_4 + K_5)\, d(x_0, y_0)$. Since $K_3 + K_4 + K_5 < 1$, it now follows that $d(x_0, y_0) = 0$; hence $x_0 = y_0$.

Theorem 2 — Suppose T is a mapping of a metric space (X, d) into itself having the property K. If there is a constant $h > 0$ such that

$$d(x, Tx) + d(y, Ty) \leqslant hd(x, y)$$

for all x, y in X, then there exists a positive integer n such that T^n is a contraction mapping.

PROOF : Since T has the property K, there exist constants $K_i (1 \leqslant i \leqslant 5)$ that satisfy (1.1) through (1.3). Now assume first that X is complete. Putting $T_1 = T_2 = T$ in the inequality (B) of Proposition 1 we get

$$d(x_n, x_{n+p}) < \frac{r^n}{1-r}\, d(x, x_1) \text{ where } x_k = T^kx \text{ and } r = \frac{K_1 + K_3 + K_5}{1 - K_2 - K_4}.$$

Letting p tend to infinity in the above inequality, we get

$$d(x_n, x_0) = d(T^nx, x_0) \leqslant \frac{r^n}{1-r}\, d(x, Tx) \text{ where } x_0 = \lim_n x_n.$$

Similarly $d(T^ny, x_0) \leqslant \frac{r^n}{1-r}\, d(y, Ty)$ and so

$$d(T^nx, T^ny) \leqslant \frac{r^n}{1-r}\, [\, d(x, Tx) + d(y, Ty)]$$

$$\leqslant \frac{r^n}{1-r}\, hd(x, y).$$

But $r^n \to 0$ as $n \to \infty$ and hence if $0 < K < 1$, then $\frac{r^n}{1-r}\, h < K$ for large n and if we fix any such n we get $d(T^nx, T^ny) \leqslant Kd(x, y)$ for all x, y, yielding that T^n is a contradiction.

Suppose now X is not complete.

$$d(Tx, Ty) \leqslant K_1d(x, Tx) + K_2d(y, Ty) + K_3d(x, Ty) + K_4d(y, Tx) + K_5d(x, y)$$

$$\leqslant d(x, Tx) + d(y, Ty) + K_3 d(x, Tx) + K_3 d(Tx, Ty) + K_4 d(y, Ty) + K_4 d(Tx, Ty) + K_5 d(x, y)$$

Hence $$(1 - K_3 - K_4)\, d(Tx, Ty) \leqslant 2d(x, Tx) + 2d(y, Ty) + K_5 d(x, y).$$
$$\leqslant (2h + K_5)\, d(x, y).$$

Therefore $$d(Tx, Ty) \leqslant \frac{2h + K_5}{1 - K_3 - K_4}\, d(x, y).$$

Thus T is uniformly continuous. Let $\tilde{X}$ be the completion of X and let $\tilde{T}$ be the completion of T. It is clear that $\tilde{T}$ has the property K. By what we have just proved, $\tilde{T}^n$ is a contraction on $\tilde{X}$ for large n. It follows that T^n is a contraction for large n on X completing the proof of the theorem.

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