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LONG WAVELENGTH PERISTALTIC TRANSPORT OF NON-NEWTONIAN FLUIDS

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Solutions are obtained for the stream function and the pressure field for the flow of non-Newtonian fluids in a tube by long peristaltic waves of arbitrary shape. The axial velocity profiles and stress distributions on the wall are discussed for particular waves of some practical interest. The effect of non-Newtonian character of the fluid is examined.

1. INTRODUCTION

The problem of peristaltic motion was studied in the past few years from the point of view of fluid mechanics by several authors. Peristaltic pumping is a kind of fluid transport which takes place by moving a wave of contraction or expansion that propagates along the length of a distensible tube containing a liquid. Particularly peristaltic pumping is often used in medical instruments such as the heart lung machine etc., and it occurs naturally in human ureters. Many authors have investigated the oscillatory flow problems in tubes considering the pressure gradient to vary exponentially. It is known that the physiological flows are not only maintained by pressure gradient but also by the motion of the boundaries. To study such situations in greater detail several investigations have been carried out to understand the various aspects of peristaltic transport of fluids (Latham and Shapiro 1966, Burns and Parkes 1967, Yin and Fung 1969, Zien and Ostrach 1970, Boyarsky *et al.* 1971). Under certain abnormal conditions or pathological conditions, the fluid in ureter can exhibit non-Newtonian character and thus the non-Newtonian effects in peristaltic motion were also included in recent studies (Kanaka Raju and Devanathan 1972, 1974; Girija Devi and Devanathan 1975; Kaimal 1978).

By introducing the concept of slowly varying cross-sections, Manton (1975) has obtained closed form solutions for the peristaltic transport of fluids with arbitrary wave shape. This analysis is more suitable for the ureteral flows since Lykoudis and Roos (1970) pointed out that the shape of the ureter during peristalsis is far from sinusoidal.

In the present work, an attempt is made to obtain the solutions for the unsteady peristaltic motion of non-Newtonian fluid in tubes of varying cross-sections by using the long wavelength approximation. It is found that the velocity field is influenced by the non-Newtonian parameter to the second order terms while the pressure distribution is modified in the first order terms. The axial velocity profiles show the presence of separation in the flow field for certain range of Reynolds numbers. From the shear stress distribution on the wall it is concluded that the maximum value of the shear stress increases due to the visco-elasticity of the fluid.

2. The Fluid Model

The rheological equation of state for the non-Newtonian visco-elastic liquids of short memory (i.e., short relaxation time) takes the simplified form

$$\sigma_{ij} = -p\delta_{ij} + t_{ij} \qquad \text{...(2.1)}$$

where

$$t_{ij} = 2\mu e_{ij} - 2k \frac{\delta}{\delta t} e_{ij} \qquad \text{...(2.2)}$$

σ_{ij} is the stress tensor, e_{ij} being the strain tensor and $\delta/\delta t$ the convected differentiation, μ is the coefficient of viscosity and k the short relaxation time. The above equation describes the equation of state for a class of liquids called the Walters liquid B'' (1962).

3. Formulation of the Problem

In order to study the peristaltic pumping in a tube of arbitrary shape, we shall consider the motion in a circular tube of varying cross-section. Let (X, R, θ) be the cylindrical polar coordinates such that $R = 0$ is the axis of symmetry of the tube and $R = a(X, t)$ is the wall of the tube. Taking (u, v) the velocity components in (X, R) directions respectively, the basic equations of motion using (2.1) and (2.2) are given by

$$\frac{\partial \bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V} = -\nabla p + \nu\nabla^2\bar{V} - k^*\Big[\frac{\partial}{\partial t}\nabla^2\bar{V} + 2(\bar{V}\cdot\nabla)\nabla^2\bar{V} - \nabla^2\{(\bar{V}\cdot\nabla)\bar{V}\}\Big] \qquad \text{...(3.1)}$$

and

$$\text{div } \bar{V} = 0$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $k^* = k/\rho$ the coefficient of elasticity, $\bar{V}$ the velocity field and p the pressure.

Boundary conditions :

$$\left.\begin{aligned} v &= \frac{\partial a}{\partial t},\ u = 0, \ \text{on}\ R = a(x, t) \\ u &= \frac{\partial u}{\partial R} = 0 \qquad \text{on}\ R = 0. \end{aligned}\right\} \qquad \ldots(3.2)$$

These boundary conditions imply that the radial motion of the fluid particles at the wall corresponds to that of the wall, and the no slip condition on the wall holds good. Also the solution is regular on the tube axis.

We shall seek a quasi-steady solution of the system (3.1) for the case when the radius of the tube varies slowly with respect to time and in the axial direction.

Equations (3.1) can be expressed in terms of vorticity field by introducing a normalized vorticity component $\omega(x, r)$ and stream function $\phi(x, r)$ by using the following scheme :

$$\left.\begin{aligned} \omega &= \left(\frac{a_0}{c}\right)\Omega, \qquad \Omega = \frac{\partial u}{\partial R} - \frac{\partial v}{\partial X} \\ u &= c\left(1 + \frac{\phi_r}{r}\right), \quad v = -c\epsilon\,\frac{\phi_x}{r} \end{aligned}\right\} \qquad \ldots(3.3)$$

$$p = \left(\frac{8\lambda\rho\nu c}{a_0^2}\right)q, \qquad \tau = \left(\frac{a_0}{4c\rho\nu}\right)T \Big\} \qquad \ldots(3.4)$$

where $r = R/a_0$ and $\epsilon = a_0/\lambda (0 < \epsilon \leqslant 1)$, r is the normalized radial coordinate and the wall profile is represented by an axisymmetric wave of constant shape propagating at speed c.

Since for a periodic wall profile the axial length scale can be taken to be the wavelength λ, the radius of the tube a is assumed to be of the form (see Manton 1975)

$$a(X - ct;\, a_0, \lambda) = a_0 s(x) \qquad \ldots(3.5)$$

where $x = (X - ct)/\lambda$ is a slowly varying normalized axial coordinate and a_0 characteristic radius of the tube.

Thus the basic equations in the dimensionless form are

$$\omega = \left(\frac{\phi_r}{r}\right)_r + \epsilon^2\left(\frac{\phi_{xx}}{r}\right) \qquad \ldots(3.6)$$

$$\text{Re}\,\epsilon\left[\phi_r\frac{\omega_x}{r} - \phi_x\left(\frac{\omega}{r}\right)_r\right] = ((r\omega)_r/r)_r + \epsilon^2\omega_{xx}$$

$$- Ke\left[\frac{\phi_r}{r}\left\{\epsilon^2\omega_{xxx} + \omega_{rrx} + \frac{\omega_{rx}}{r} - \frac{\omega_x}{r^2}\right\} - \right.$$

(*equation continued on p.* 510)

$$- \frac{\phi_x}{r} \left\{ \epsilon^2(\omega_{xxr} + \omega_{xx}) + \omega_{rrr} + 2\,\frac{\omega_{rr}}{r} - \frac{9}{r}\left(\frac{\omega}{r}\right)_r \right\}$$

$$- 4\,\frac{\phi_{rx}}{r}\left(\frac{\omega}{r}\right)_r - 4\epsilon^2\phi_{xx}\,\frac{\omega_x}{r^2}\Bigg] \qquad \ldots(3.7)$$

where the subscripts x and r denote partial differentiation with respect to these variables.

The resulting non-dimensional parameters are

$$\left.\begin{aligned} Re &= \frac{a_0 c}{\nu} = \text{Reynolds number} \\ Ke &= \frac{k^* c}{\nu\lambda} = \text{non-Newtonian parameter.} \end{aligned}\right\} \qquad \ldots(3.8)$$

The corresponding boundary conditions are:

$$\phi = -\beta/2,\ \phi_r = -s, \text{ on } r = s(x) \qquad \ldots(3.9)$$

$$\phi,\ \phi_x/r,\ (\phi_r/r)_r \to 0 \text{ as } r \to 0 \qquad \ldots(3.10)$$

where β is related to the mean volume flux α through the tube in such a way that

$$\beta = 1 - \alpha. \qquad \ldots(3.11)$$

We shall obtain solution of (3.6) - (3.10) for the velocity and pressure by using the long wavelength approximation as in Manton (1975). Considering typical values of λ and r for the ureteral flow, it is found that $\lambda \gg a$, hence the long wavelength approximation is well justified for the present case.

Thus taking ϕ, ω and q in powers of ϵ, we have

$$\left.\begin{aligned} \phi &= \sum_n \epsilon^n \phi^{(n)}(x, r) \\ \omega &= \sum_n \epsilon^n \omega^{(n)}(x, r) \\ q &= \left(\frac{a_0^2 p}{8\lambda\rho\nu c}\right),\ p = \sum_n \epsilon^n p^{(n)}(x, r). \end{aligned}\right\} \qquad \ldots(3.12)$$

Substituting (3.12) in (3.6) - (3.10) and separating the terms of various orders of ϵ (by taking Ke as $O(\epsilon)$) we can obtain a system of coupled equations for $\phi^{(0)}$, $\phi^{(1)}$, $\omega^{(0)}$, $\omega^{(1)}$ etc. Solving these equations under the boundary conditions as specified earlier, the final solutions for stream function and vorticity are given by :

$$\Phi = \tfrac{1}{2}(\beta - s^2)\,\eta^4 - (\beta - \tfrac{1}{2}s^2)\,\eta^2 + \frac{\epsilon Re}{36s}\,\eta^2 \frac{ds}{dx}(\beta - \tfrac{1}{2}s^2)$$

$$\times \{(\beta - s^2)\,\eta^6 - 6(\beta - \tfrac{1}{2}s^2)\,\eta^4 + 9(\beta - \tfrac{1}{3}s^2)\,\eta^2 - 4(\beta - \tfrac{1}{4}s^2)\}$$

$$+ \frac{\epsilon^2\eta^2}{12}\left[\left(\frac{ds}{dx}\right)^2 \left\{(\eta^2 - 1)^2(3s^2 - 10\beta) + \frac{Re^2}{21600}(a_1\beta^3 - a_2\beta^2\right.\right.$$

$$\left. + a_3\beta - a_4) + \frac{KRe}{30}(a_5\beta^3 - a_6\beta^2 + a_7\beta - a_8)\right\}$$

$$+ \frac{d^2s}{dx^2}\left\{(\eta^2 - 1)^2(2\beta s - 1) - \frac{Re^2}{21600}(b_1\beta^3 - b_2\beta^2 + b_3\beta - b_4)\right.$$

$$\left.\left. - \frac{KRe}{30}(b_5\beta^3 - b_6\beta^2 + b_7\beta - b_8)\right\}\right] + O(\epsilon^3) \qquad \text{...(3.13)}$$

and

$$\omega = \frac{4}{s^3}(\beta - s^2)\,\eta - \frac{2\epsilon Re}{3s^4}\,\eta \frac{ds}{dx}(\beta - \tfrac{1}{2}s^2)\{2(\beta - s^2)\,\eta^4$$

$$- 6(\beta - \tfrac{1}{2}s^2)\,\eta^2 + 3(\beta - \tfrac{1}{3}s^2)\} + \frac{\epsilon^2\eta}{3}$$

$$\times\left[\left(\frac{ds}{dx}\right)^2 \times \left\{\beta s^{-3}(22 - 30\eta^2) + 3s^{-1}(3\eta^4 - 4)\right.\right.$$

$$\left. + \frac{Re^2}{720}(c_1\beta^3 - c_2\beta^2 + c_3\beta - c_4) + \frac{KRe}{45}(c_5\beta^3 - c_6\beta^2 + c_7\beta - c_8)\right\}$$

$$+ \frac{d^2s}{dx^2}\left\{2\beta s^{-2}(3\eta^2 - 1) - (3\eta^2 - 4) - \frac{Re^2}{720}(d_1\beta^3 - d_2\beta^2 + d_3\beta - d_4)\right.$$

$$\left.\left. - \frac{KRe}{45}(d_5\beta^3 - d_6\beta^2 + d_7\beta - d_8)\right\}\right] + O(\epsilon^3). \qquad \text{...(3.14)}$$

where $\eta = r/s$, $a_1 \ldots a_8$, $b_1 \ldots b_8$, $c_1 \ldots c_8$ and $d_1 \ldots d_8$ are polynomials in η which are given in Appendix.

It can be clearly seen that the solutions (3.13) and (3.14) with $K = 0$ coincides with the solutions for Newtonian fluids as obtained in Manton (1975) though the mode of expansions in the present case is slightly different.

The simplified form of pressure distribution is given by

$$q(x, r) = \int^{x} (\beta - s^2)\frac{dx}{s^4} - \frac{\epsilon}{24}\, s^{-4}\{Re(3\beta^2 - \beta s^2 + 2s^3 \ln s)$$

$$- 8Ks^{-2}(4\beta^2 - 6\beta s^2 - 3s^4)\} +$$

(equation continued on p. 512)

$$+ \epsilon^2 \Big[\frac{s^{-3}}{2880}\frac{ds}{dx}\{480(\beta(6\eta^2 - 1) - 2\eta^2 s^2)$$
$$+ Re^2 s^{-2}(44\beta^3 - 62\beta^2 s^2 + 26\beta s^4 - 3s^6)$$
$$- 48KRe\, s^{-4}(12\beta^3 - 20\beta^2 s^2 + 9\beta s^4 - s^6)\}$$
$$+ \frac{1}{2160}\int^{x} \{2880 s^2(\beta - \tfrac{1}{2}s^2) + Re^2(44\beta^3 - 82\beta^2 s^2$$
$$+ 40\beta s^4 - 5s^6) + 72KRe(12\beta^2 - 8\beta s^2 + s^4)\}$$
$$\times \left(\frac{ds}{dx}\right)^2 \frac{dx}{s^6}\Big]. \qquad \ldots(3.15)$$

It is noticed that the contribution of the non-Newtonian terms on the velocity and vorticity fields is seen in the second order solutions, while the pressure field influenced in the first order solution itself, since $Ke = K\epsilon$.

Taking only the leading term in (3.15) it is found that the maximum and minimum pressures occurs at position where $s^2 = 1 - \alpha + O(\epsilon^2)$, i.e., in the contracted region of the tube. Further the non-dimensional mean pressure gradient can be written as

$$\gamma = q(1, r) - q(0, r) \qquad \ldots(3.16)$$

and using (3.15) and (3.16) we get

$$\gamma = \int_0^1 \frac{dx}{s^4}(\beta - s^2) + \epsilon^2 \int_0^1 \{2880(\beta - \tfrac{1}{2}s^2)$$
$$+ Re^2 s^{-2}(44\beta^3 - 82\beta^2 s^2 + 40\beta s^4 - 5s^6)$$
$$+ 72KRe\, s^{-2}(12\beta^2 - 8\beta s^2 + s^4)\}\left(\frac{ds}{dx}\right)^2 \frac{dx}{s^4}. \qquad \ldots(3.17)$$

It is evident that for a peristaltic pump, we must have $\alpha \geqslant 0$, and $\gamma \geqslant 0$.

From eqns. (3.11) and (3.17) correct to order of ϵ we have

$$s^2_{min} < \beta \leqslant 1, \text{ or } 0 \leqslant \alpha < 1 - s^2_{min} < 1 \qquad \ldots(3.18)$$

The radial force per unit area acting on the fluid at the wall $R = a$ is

$$F = -\left(\sigma_{rr} - \sigma_{xr}\,\epsilon\frac{ds}{dx}\right)\Big/\left\{1 + \epsilon^2\left(\frac{ds}{dx}\right)^2\right\}^{1/2} \qquad \ldots(3.19)$$

where the stress tensor is given by (2.1). From (3.3), (3.6), (3.9) and (3.19) we get

$$F = \left\{p - \epsilon\rho\nu(\Omega + 2c/a)\frac{ds}{dx}\right\}\Big/\left\{1 + \epsilon^2\left(\frac{ds}{dx}\right)^2\right\}^{1/2}. \qquad \ldots(3.20)$$

The net rate of working of the wall over a wavelength is

$$W_0 = \int_0^{\lambda} 2\pi a F v \, dx. \qquad \text{...(3.21)}$$

The normalized rate of working is given by

$$\delta = W_0/8\pi\rho\nu c^2\lambda. \qquad \text{...(3.22)}$$

We find from (3.2), (3.5), (3.16) and (3.19) - (3.22)

$$\delta = 1 + \gamma - \beta \int_0^1 \frac{dx}{s^2} + \frac{\epsilon^2}{360}\left[\int_0^1 s^{-4}\{60s^2(s^2 - 2\beta) - Re^2(44\beta^3 - 82\beta^2 s^2 + 40\beta s^4 - 5s^6) - 12KRe(12\beta^2 - 8\beta s^2 + s^4)\}\left(\frac{ds}{dx}\right)^2 dx + 360\int_0^1 s dx \left(\frac{ds}{dx}\right)^3 \int^x \frac{dx}{s^4}(\beta - s^2)\right] \qquad \text{...(3.23)}$$

Equations (3.17) and (3.23) show that the net rate of working of the wall is linearly increasing function of the mean pressure gradient γ and mean flux α and is modified by the parameter K.

4. Shear Stress at the Wall

The non-Newtonian character is predominant in the stress distribution. The shear stress at the wall of the tube $R = a(x, t)$ is

$$T = (\sigma_{rr} - \sigma_{xx})\,\epsilon\,\frac{ds}{dx} + \sigma_{xr}\left\{1 - \epsilon^2\left(\frac{ds}{dx}\right)^2\right\}\Big/\left\{1 + \epsilon^2\left(\frac{ds}{dx}\right)^2\right\} \qquad \text{...(4.1)}$$

where σ_{xr}, σ_{xx}, σ_{rr} are stress components with usual meaning. Using (3.3), (3.6), (3.9) the shear stress distribution simplifies to

$$T = \rho\nu\Omega + \frac{\rho\nu c\epsilon^2}{a}\left[4\left(\frac{ds}{dx}\right)^2 + 2s\frac{d^2s}{dx^2} - \epsilon^2\left(\frac{ds}{dx}\right)^2\left\{\left(\frac{ds}{dx}\right)^2 + s\frac{d^2s}{dx^2}\right\}\right]\Big/\left\{1 + \epsilon^2\left(\frac{ds}{dx}\right)^2\right\}. \qquad \text{...(4.2)}$$

The first term is the shear stress at the wall of a fixed tube and the second term is due to finite radial velocity at the wall.

Substituting the solution (3.14) for Ω in (4.2) we have

$$\tau = (\beta/s^3 - 1/s) + \frac{\epsilon Re}{6s^4}\frac{ds}{dx}\beta(\beta - \tfrac{1}{2}s^2) + \frac{\epsilon^2}{540s^7}\Big[\left(\frac{ds}{dx}\right)^2$$

$$\times \Big\{45s^4(21s^2 - 20s^4) + \frac{Re^2s^2}{4}(1682\beta^3 - 110\beta^2s^2 + 28\beta s^4 - 13s^6)$$

$$+ 4KRe(756\beta^3 - 1178\beta^2s^2 + 1744\beta s^4 - 9s^6)\Big\} + \frac{d^2s}{dx^2}\Big\{3s^5(60\beta$$

$$+ 105s^2) + \frac{Re^2}{4}s^3(960\beta^3 - 66\beta^2s^2 + 24\beta s^4 - 3s^6)$$

$$+ 4KRes(1188\beta^3 - 90\beta^2s^2 + 36\beta s^4 - 9s^6)\Big\}\Big] \quad ...(4.3)$$

where τ is normalized shear stress defined in (3.4).

Taking the dominant terms in (4.3) the inequalities giving the upper and lower bounds on the shear stress can be derived and they happen to be same as in Manton (1975).

5. Discussion

The peristaltic motion of a non-Newtonian fluid in a tube of slowly varying cross-section in the axial direction has been investigated under the long wavelength approximations. The streamfunction, vorticity and pressure distributions are obtained as power series in terms of the amplitude of deformation. The solutions are very much modified due to non-Newtonian property of the fluid. The solutions are valid for any arbitrary shape of the wave. By prescribing a suitable functional relation for s, the solutions can be simplified to a great extent. For instance Lykoudis and Roos (1970) have taken a model for ureter in which the wave shape is:

$$s(x) \begin{aligned} &= B + Ax^n \quad 0 < x < X \\ &= B + AX^n \quad X < x < 1 \end{aligned} \quad ...(5.1)$$

where $A = 6.9$, $B = 0.014$, $X = 0.69$, $n = 4$, $\lambda = 36$ cm, and $a_0 = 0.15$ cm.

Taking the above particular shape for the peristaltic wave, the shear stress distribution is calculated and shown in Fig. 1 for Newtonian and non-Newtonian fluids. It is clear that the vanishing of the shear stress on the axis can be interpreted as the existence of separation in the flow field. It is seen that there is only one point of separation for Newtonian fluid while the fluid separates twice for non-Newtonian fluids. Moreover, it is interesting to see that the maximum value of the shear stress increases for non-Newtonian fluids as compared to Newtonian fluids. Even the minimum value of shear stress is greatly modified due to the

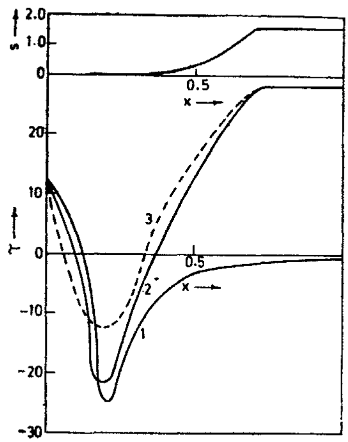

FIG. 1. Distribution of wall shear stress τ for the wave shape $s(x)$ with $\beta = 2.3 \times 10^{-4}$, $n = 4$, $A = 6.9$, $B = 0.014$, $X = 0.69$. (1) $Ke = 0.0$, (2) $Ke = 0.05$, (3) $Ke = 0.8$.

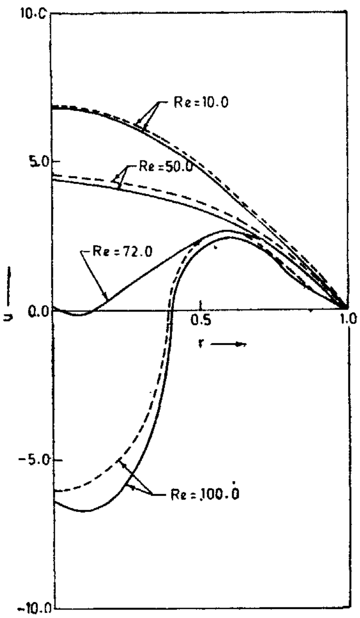

FIG. 2. Axial velocity profiles for different Reynolds number ($Ke = 0.0$ ——, $Ke = 0.5$ - - - -).

presence of non-Newtonian character of the fluid. Such estimates of the value of shear stress might be of some use in analysing the properties of abnormal human ureters.

Once again taking the shape of the wave as (5.1) the axial velocity profiles are depicted graphically in Fig. 2 for different Reynolds number. The profiles for Newtonian and non-Newtonian fluids show that they remain positive for certain range of Reynold numbers and they start from negative values and attain the value 1 on the axis. From the computation of the solutions it is found that the profiles remain positive for all Reynolds numbers less than 70 and remain negative and positive for $Re = 100$ and completely negative for $Re > 365$. Such a behaviour in the axial velocity profiles can be physically explained as the case when the fluid motion is opposite to the peristaltic wave. In fact, the negative axial velocity is might be due to the occurrence of the phenomenon of 'reflux' which takes place in certain abnormal conditions such as in hydroureters. It is found that the critical Reynolds numbers for the change of axial velocity profiles from positive to negative or for remaining completely negative depends on the value of the non-Newtonian parameter. Thus the presence of non-Newtonian character influences the onset of reflux phenomenon.

Although the above discussions for shear stress distribution and the axial velocity profiles are carried out for the model of the wave shape given by Lykoudis and Roos (1970), it is concluded that the computation of these profiles for any wave shape can be carried out and the behaviour can be studied in detail with the help of the general solutions obtained in the earlier section.

REFERENCES

Boyarsky, S., Gottschalk, C. W., Tanagho, E. A., and Zimskind, P. D. (1971). Urodynamics. Academic Press, New York.

Burns, J. C., and Parkes, T. (1967). Peristaltic motion. *J. Fluid Mech.*, **29**, 731.

Girija Devi, R., and Devanathan, R. (1975). Peristaltic motion of a micropolar fluid. *Proc. Indian Acad Sci.*, **81A**, 149.

Kanaka Raju, K., and Devanathan, R. (1972). Peristaltic motion of a non-Newtonian fluid. *Rheol. Acta.*, **11**, 170.

——————— (1974). Peristaltic motion of a non-Newtonian fluid—II. *Rheol. Acta*, **13**, 944.

Kaimal, M. R. (1978). Peristaltic pumping of non-Newtonian fluid with particles suspended in it at low Reynolds number under long wave approximations. *J. appl. Mech.*, **45**, 32.

Latham, T. W., and Shapiro, A. H. (1966). Fluid motions in a peristaltic pump. *Proc. Ann. Conf. Engg. Med. Biol.*, **8**, 147.

Lykoudis, P. S., and Ross, R. (1970). The fluid mechanics of the ureter from a lubrication point of view. *J. Fluid Mech.*, **43**, 661.

Manton, M. J. (1975). Long wave-length peristaltic pumping at low Reynolds number. *J. Fluid Mech.*, **68**, 467.

Walters, K. (1962). Non-Newtonian effects in some elasticoviscous liquids. *Q. Jl appl. Math. Mech.*, **15**, 63.

Yin, F., and Fung, Y. C. (1969). Peristaltic waves in circular cylindrical tubes. *J. appl. Mech.*, **37**, 579.

Zien, T. F., and Ostrach, S. (1970). A long wave approximation to peristaltic motion. *J. Biomech.*, **3**, 63.

APPENDIX

The coefficients $a_1, a_2, \ldots d_8$ given in the expressions (3.13) and (3.14) are recorded below :

$$a_1 = 12s^{-2}(38\eta^{10} - 390\eta^8 + 1425\eta^6 - 2650\eta^4 + 2395\eta^2 - 818)$$
$$a_2 = 30(34\eta^{10} - 270\eta^8 + 775\eta^6 - 1100\eta^4 + 785\eta^2 - 224)$$
$$a_3 = 6s^2(116\eta^{10} - 690\eta^8 + 1575\eta^6 - 1675\eta^4 + 805\eta^2 - 131)$$
$$a_4 = 3s^4(44\eta^{10} - 210\eta^8 + 375\eta^6 - 250\eta^4 + 46\eta^2 - 5)$$
$$a_5 = 2s^{-4}(11\eta^8 - 80\eta^6 + 300\eta^4 - 404\eta^2 + 173)$$
$$a_6 = s^{-2}(53\eta^8 - 300\eta^6 + 780\eta^4 - 872\eta^2 + 339)$$
$$a_7 = 2(19\eta^8 - 85\eta^6 + 165\eta^4 - 151\eta^2 + 52)$$
$$a_8 = s^2(7\eta^8 - 30\eta^6 + 45\eta^4 - 28\eta^2 + 6)$$

$$b_1 = 36s^{-1}(2\eta^{10} - 20\eta^8 + 75\eta^6 - 150\eta^4 + 145\eta^2 - 52)$$
$$b_2 = 30s(6\eta^{10} - 48\eta^8 + 145\eta^6 - 240\eta^4 + 207\eta^2 - 70)$$
$$b_3 = 18s^3(8\eta^{10} - 50\eta^8 + 125\eta^6 - 175\eta^4 + 135\eta^2 - 43)$$
$$b_4 = 3s^5(12\eta^{10} - 60\eta^8 + 125\eta^6 - 150\eta^4 + 105\eta^2 - 32)$$
$$b_5 = 2s^{-3}(3\eta^8 - 20\eta^6 + 60\eta^4 - 72\eta^2 + 29)$$
$$b_6 = 5s^{-1}(3\eta^8 - 16\eta^6 + 36\eta^4 - 36\eta^2 + 13)$$
$$b_7 = 2s(6\eta^8 - 25\eta^6 + 45\eta^4 - 39\eta^2 + 13)$$
$$b_8 = s^3(3\eta^8 - 10\eta^6 + 15\eta^4 - 2\eta^2 + 4)$$

$$c_1 = 2s^{-5}(228\eta^8 - 585\eta^6 + 3420\eta^4 - 3180\eta^2 + 958)$$
$$c_2 = 10s^{-3}(102\eta^8 - 540\eta^6 + 930\eta^4 - 660\eta^2 + 157)$$
$$c_3 = 2s^{-1}(348\eta^8 - 1380\eta^6 + 1890\eta^4 - 1005\eta^2 + 161)$$
$$c_4 = s(132\eta^8 - 420\eta^6 + 450\eta^4 - 150\eta^2 + 1)$$
$$c_5 = 12s^{-7}(55\eta^6 - 240\eta^4 + 450\eta^2 - 202)$$
$$c_6 = 6s^{-5}(265\eta^6 - 900\eta^4 + 1170\eta^2 - 872)$$
$$c_7 = 6s^{-3}(190\eta^6 - 510\eta^4 + 495\eta^2 - 151)$$
$$c_8 = 3s^{-1}(70\eta^6 - 180\eta^4 + 135\eta^2 - 28)$$

$$d_1 = 12s^{-4}(6\eta^8 - 40\eta^6 + 15\eta^4 - 90\eta^2 + 29)$$
$$d_2 = 6s^{-2}(30\eta^8 - 160\eta^6 + 290\eta^4 - 240\eta^2 + 69)$$
$$d_3 = 6(24\eta^8 - 100\eta^6 + 150\eta^4 - 105\eta^2 + 27)$$
$$d_4 = 3s^2(12\eta^8 - 40\eta^6 + 50\eta^4 - 30\eta^2 + 7)$$
$$d_5 = 36s^{-6}(5\eta^6 - 20\eta^4 + 30\eta^2 - 48)$$
$$d_6 = 90s^{-4}(5\eta^6 - 16\eta^4 + 18\eta^2 - 6)$$
$$d_7 = 18s^{-2}(20\eta^6 - 50\eta^4 + 45\eta^2 - 13)$$
$$d_8 = 9(10\eta^6 - 20\eta^4 + 15\eta^2 - 4).$$