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A NOTE ON SIMILARITY OF OPERATORS AND RELATED RESULTS

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Some theorems of Khasbardar and Thakare (1978) on similarity of operators involving their inverses and adjoints are extended for the case of essential similarity.

1. Introduction

The object of the present exposition is to extend certain results of Khasbardar and Thakare (1978) on similarity of operators involving their inverses and adjoints. The results concern about essential similarity of such operators. First we give some definitions. Let $B(H)$ be the Banach algebra of all bounded linear operators on a Hilbert Space H. Denote by $\sigma(T)$ the spectrum of T and $W(T)$ the numerical range of T. $\partial\sigma(T)$ will stand for the boundary of $\sigma(T)$ and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. $\overline{W(T)}$ will stand for the closure of the numerical range. $r(T)$ and $w(T)$ will denote the spectral radius and the numerical radius of T respectively. Let $BC(H)$ denote the ideal of compact operators on H. Then $B(H)/BC(H)$ is called the Calkin algebra. Let $\hat{T}$ denote the canonical image of T in the Calkin algebra. $\sigma(\hat{T})\,(=\sigma_e(T))$ and $W(\hat{T})\,(=W_e(T))$ are known as essential spectrum and the essential numerical range of T respectively. As shown in Fillmore *et al.* (1972, Theorem 9) $W_e(\hat{T}) = \bigcap\limits_K \overline{W(T+K)}$, the intersection is taken all over compact operators K.

An operator T is called normaloid if $w(T) = \| T \|$, Convexoid if $\overline{W(T)} = \text{conv}\ \sigma(T)$ (conv. denotes the convex hull) and spectraloid if $r(T) = w(T)$. An operator T will be called essentially normaloid essentially convexoid or essentially spectraloid if $\hat{T}$ is normaloid, convexoid or spectraloid respectively. The first result given below gives a result anologous to Theorem 2.1 of Khasbardar-Thakare (1978) for operators whose inverse is essentially similar to it's adjoint.

2. Results

Theorem 2.1 — Let T, $S \in B(H)$ with T invertible. If

(i) $T^{-1}S = ST^* + K$, with $0 \notin \overline{W_e(S)}$ and K compact

(ii) $\pi_{00}(T) = \phi$ and $\pi_{00}(T^{-1}) = \phi$

(iii) both T and T^{-1} are spectraloid, then T is unitary.

PROOF : Because of conditions (i) and (ii), by Corollary 2 of Patel (1974b) $\sigma(T) \subset$ unit disc Δ. T being spectraloid, $w(T) = r(T) = 1$ gives us $W(T) \subset \Delta$. Similar arguments for T gives $W(T^{-1}) \subset \Delta$. Now the theorem follows from a result of Stampfli (1967).

Theorem 2.2 — Let T be an operator such that $T - \alpha$ is spectraloid for every complex α. If $ST = T^*S + K$ for an operator S with $0 \notin \overline{W_e(S)}$, for K compact and $\pi_{00}(T) = \phi$, then T is self-adjoint.

PROOF : By Furuta-Nakamoto's Lemma (1971), T is convexoid. Also under the given conditions, $\sigma(T)$ is real by Corollary 1 of Patel (1974b). Thus T is self-adjoint.

Let us call the condition (iii) of Theorem 2.1 as (G_1) condition. If any direct summond of T satisfies the condition (G_1), then T will be called reduction-(G_1) operator. We have the following theorem about a reduction-(G_1) operator. Note that this definition of reduction-(G_1) operator differs from that of Patel (1974b).

Theorem 2.3 — Let $T, S \in B(H)$ with T invertible. If

(i) $T^{-1} S = ST^* + K$ with $0 \notin \overline{W_e(S)}$ and K compact,

(ii) T is reduction-(G_1), then T is normal.

PROOF : Let M be the closed subspace defined as the closed linear span of all reducing eigenspaces of the operator T. Then the restriction of T to M denoted by $T_1 = T \mid M$ is normal. Let $T_2 = T \mid M^{\perp}$, where $M^{\perp}$ stands for the orthogonal complement of M. Then $T = T_1 \oplus T_2$. First of all we claim that $\pi_{00}(T_2) = \phi$ and to show that the argument is same as in Patel (1974b). Similarly it can be shown that $\pi_{00}(T_2^{-1}) = \phi$. Now applying theorem 2.1 of our present note since T is reduction-(G_1) we get T_2 as unitary operator and hence T is normal.

Following two theorems are analogues of Theorems 2.4 and 2.5 of Khasbardar-Thakare (1978) concerning essential similarity.

Theorem 2.4 — If $T - \alpha$ is spectraloid for all complex number α and if

$$T^{-1}S = ST^* + K,\ 0 \notin \overline{W_e(S)}$$

and K compact, and also $\pi_{00}(T)$ is empty, then T is unitary if $w(T) = w(T^{-1})$.

Theorem 2.5 — If T is an invertible spectraloid operator which is similar to a unitary plus a compact operator and suppose that $\pi_{00}(T) = \phi$, also if T^{-1} is a contraction, then T is unitary.

We omit the proofs of the above theorems because they follow our earlier two theorems.

Our next theorem stems from Theorem 4.1 of Khasbardar-Thakare (1978) where the similarity operator involved is a normal operator.

Theorem 2.6 — Let T be an invertible operator. If there exists a normal operator S with polar decomposition UP and $T^* = S^{-1}T^{-1}S + K$, $0 \notin \overline{W_e(S)}$, then T is similar to a unitary operator plus a compact operator if U commutes with T.

PROOF : We have $T^* = P^{-1}U^*T^{-1}UP + K$. By hypothesis P is positive and invertible and $P^{1/2}$ will be it's positive square root. Let $A = P^{-1/2}TP^{1/2}/(*)$. We will prove that A is similar to a unitary operator plus a compact operator [In the following $K_i(i = 1, 3, 4, 5)$, will stand for compact operators]. Now

$$\begin{aligned} A^* = P^{-1/2}T^*P^{-1/2} &= P^{1/2}P^{-1}U^*T^{-1}UPP^{-1/2} + K_1 \\ &= P^{-1/2}U^*T^{-1}UP^{1/2} + K_1 \\ &= P^{-1/2}T^{-1}P^{1/2} + K_1 \text{ (since } UP = PU \text{ and } UT = TU) \end{aligned}$$

i.e., $A^* = A^{-1} + K_1$

i.e., $A^*A = I + K_3$, where K_3 is another compact operator. From which it follows that $(A^*A)^{1/2} = I + K_4$.

If $A = VQ$ is the polar decomposition of the operator A, then it is clear that V is unitary and $Q = (A^*A)^{1/2}$. Also $A = V[I + K_4] = V + K_5$. Hence from (*) the result follows.

An operator T is called Fredholm if T is invertible in the Calkin algebra. We have the following theorem for essentially normal Fredholm operator.

Theorem 2.7 — Let T be an essentially normal Fredholm operator and let

$$\hat{T}^* = \hat{S}^{-1}\hat{T}^{-k}\hat{S}$$

for a Fredholm operator $S \in B(H)$ such that $0 \notin \overline{W(\hat{S})}$ and $k \geqslant 2$. Then $\hat{T}^{k-1}$ is a projection operator.

PROOF : Following the arguments of Khasbardar-Thakare (1978) it can finally be shown that $\hat{T} = \hat{T}^k$ because of the conditions stated in the hypothesis. Hence $\hat{T}^{k-1}$ is a projection (see Furuta and Nakamoto 1971).

Remark 1 : Lastly we want to mention that Theorem 2.3 of Khasbardar-Thakare (1978) can also be derived as a consequence of Corollary 1 of Theorem 2 of Stampfli (1967) because of the fact that we have an equivalence result of the following nature from Salinas (1973).

Lemma (Salinas 1973) — Let $T \in B(H)$. Then following conditions are equivalent :

(i) T is convexoid;

(ii) $\| (T - \beta)^{-1} \| \leqq 1/d(\beta, \Sigma(T))$
where $\Sigma(T) = \text{conv}\, \sigma(T)$ and $\beta \notin \Sigma(T)$;

(iii) $r(T + \beta) = w(T + \beta)$ for all complex β;

(iv) $w((T - \beta)^{-1}) \leqq d(\beta, \Sigma(T))$ for $\beta \notin \Sigma(T)$.

Remark 2 : We note that Furuta-Nakamoto's result also comes from the equivalence of (i) and (iii) of the above lemma.

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