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ON CYCLES OF k INTEGERS

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If more than two nonnegative integers

$$a_1, a_2, a_3, \ldots, a_k$$

not all zero, are arranged round a circle in this order, we get what we call a k-cycle and write

$$(a_1, a_2, a_3, \ldots, a_{k-1}, a_k).$$

Such a cycle has a unique successor given by the cycle

$$(\,|a_1 - a_2|,\ |a_2 - a_3|,\ \ldots,\ |a_{k-1} - a_k|,\ |a_k - a_1|\,)$$

except when the a's are all equal and the cycle has no successor. A cycle may or may not, however, have a predecessor. In this paper, problems arising from such considerations are discussed from different angles. An application to graph theory is also given.

1. INTRODUCTION

Let k be any fixed integer > 2. Take k nonnegative integers

$$a_1, a_2, \ldots, a_k$$

'not all zero', and arrange them in this order round a circle at equal intervals preferably, it being immaterial whether they are arranged in the clock-wise or the anti-clock-wise direction. Then the figure so obtained will represent the k-cycle

$$A = (a_1, a_2, \ldots, a_k). \qquad \ldots(1.1)$$

The set of all such cycles will be denoted by C_k.

Due to the distinct positions of the elements in (1.1), even equal elements will be regarded as distinct.

The cycle with $a_1 = a_2 = \ldots = a_k = 1$, will be denoted invariably by J.

A cycle will be said to be primitive if the g.c.d. of its elements is 1. If the g.c.d. of the elements of a cycle A is $g > 1$, then we shall write A also in the form:

$$A = g(a_1/g, a_2/g, \ldots, a_k/g). \qquad \ldots(1.2)$$

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In what follows, A shall denote a cycle which is not a multiple of J, unless stated otherwise. This shall apply to other such cycles also.

Corresponding to the cycle in (1.1), we have the row matrix

$$A_{1\times k} = [a_1, a_2, \ldots, a_k] \qquad \ldots(1.3)$$

of which the elements are the same as those of A and appear in the same order as they do in the cycle A.

The transpose of $A_{1\times k}$ will be denoted for simplicity by $A_{k\times 1}$.

Again, taking $A(a_1, a_2, \ldots, a_k)$ as a lattice-point in the k-dimensional orthogonal Euclidean space, we shall use this point to represent the cycle A itself.

Recall that the axes $X_1'OX_1, X_2'OX_2, \ldots, X_k'OX_k$ in the k-dimensional orthogonal Euclidean space divide it into 2^k compartments (octants for $k = 3$). In each of these compartments, there is a point with each of its coordinates equal to 1 in magnitude but not necessarily with a plus sign. To distinguish between these points we will have to label the compartments with a number assigned to each. This we do as follows. We explain with the help of an example.

In the six-dimensional space, consider the point $(1, -1, -1, 1, -1, 1)$. Replacing each 1 by 0 and each -1 by a 1, we get the six-digited number

011010.

In the binary system this represents the number 26 and we allot the number 27 to the compartment in which the said point lies. The point $(1, -1, -1, 1, -1, 1)$ can now be very conveniently represented by $J^{(27)}$.

In particular, we shall denote the point $(1, 1, 1, 1, 1, 1)$ by J in place of $J^{(1)}$.

Our scheme is not in conformity with the usual convention in the two-dimensional case but it is certainly more scientific.

When we are not specifically concerned with the compartment in which such a point lies but it is not in compartment 1, we shall denote it by J^*.

2. System of Cycles Generated by a Given Cycle

Let

$$A = (a_1, a_2, \ldots, a_k)$$

be the given cycle. Then it generates the doubly infinite system of cycles given by

$$G(A, x, y) = (x + a_1 y, x + a_2 y, \ldots, x + a_k y) \qquad \ldots(2.1)$$

where except that $y \neq 0$, x and y run over all integers for which G continues to belong to C_k. In the matrix notation, we can write

$$G(x, y) = G(A, x, y) = (xJ + yA). \qquad \ldots(2.2)$$

Here we have written J for $J_{1\times k}$ and A for $A_{1\times k}$.

Two members of the set $G(x, y)$ are of particular importance to us. These are obtained if we take

(i) $x = -\min(a_1; a_2; \ldots; a_k) = -a_m$ (say) and $y = 1$;

(ii) $x = \max(a_1; a_2; \ldots; a_k) = a_h$ (say) and $y = -1$.

Writing explicitly, these are

$$\text{(i)} \quad M = M(A) = (a_1 - a_m, a_2 - a_m, \ldots, a_k - a_m); \qquad \ldots(2.3)$$

$$\text{(ii)} \quad H = H(A) = (a_h - a_1, a_h - a_2, \ldots, a_h - a_k). \qquad \ldots(2.4)$$

The symbols M and H will be reserved for these members throughout.

Note that each of these two cycles has at least one element equal to zero. Moreover, the sum of the corresponding elements in the two remains fixed for any given A. Here it is $a_h - a_m$. Each of the two cycles will, therefore, be said to be the complement of the other.

Graphically, the system generated by A can be represented as follows:

We shall assume that all lines in the following are produced indefinitely in both directions and O represents as usual the origin of the system of coordinates in the k-dimensional orthogonal Euclidean space. A and J represent the points whose coordinates are

$$(a_1, a_2, \ldots, a_k) \text{ and } (1, 1, 1, \ldots, 1).$$

respectively.

On the lines OJ and OA mark the lattice-points

$$\ldots -xJ, \ldots, -3J, -2J, -J, O, J, 2J, 3J, \ldots, xJ, \ldots, x \geqslant 0;$$

$$\ldots -yA, \ldots, -3A, -2A, -A, O, A, 2A, 3A, \ldots, yA, \ldots, y \geqslant 0.$$

Through each of the marked points on OJ draw lines parallel to OA, and through each of the marked points on OA draw lines parallel to OJ. This gives us a coordinate system (not necessarily orthogonal) where we can conveniently call OJ as the axis of J and OA as the axis of A. The point (x, y) where the lines through xJ and yA intersect, then represents the cycle $(xJ + yA)$. The aggregate of points of intersection will represent the cycles of the system generated by A, if we exclude from these the points on OJ and also those for which $xJ + yA$ has any coordinate negative.

Evidently x and y cannot both be negative.

For y positive,

$$x + ya_i \geqslant 0, \text{ for } 1 \leqslant i \leqslant k,$$

if and only if

$$x + ya_m \geqslant 0, \text{ i.e. } x \geqslant -ya_m; \qquad \ldots(2.5)$$

for y negative,

$$x + ya_i \geqslant 0, \text{ for } 1 \leqslant i \leqslant k,$$

if and only if

$$x + ya_h \geqslant 0, \text{ i.e. } x \geqslant -ya_h. \qquad \ldots(2.6)$$

It will be readily seen that in our graph (Fig. 1) M is the point $(-a_m, 1)$ and H is the point $(a_h, -1)$. Hence all points of intersection within the angle HOM other than those on OJ, represent the acceptable cycles of the system generated by A and there are no others.

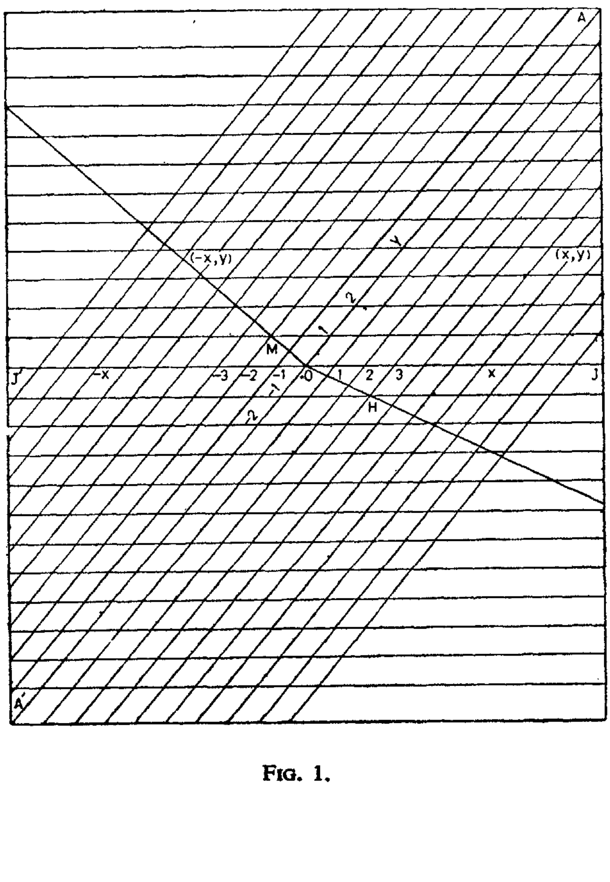

Fig. 1.

We shall speak of the acceptable members of $G(A, x, y)$ as the collaterals of A and of A as their generator.

Note that J generates only its own multiples, and for these $y = 0$.

Theorem 1 — M generates the same system exactly as does A. And this is equally true of H.

We prove the theorem for M. The proof for H is similar and is left to the reader.

PROOF : Let (X, Y) be the coordinates of $P(x, y)$ with reference to the axes OJ and OM. Then, M generates the system

$$XJ + YM, \quad Y \neq 0, \text{ and } X, Y \text{ suitable integers.}$$

Since $M = -a_m J + A$, we have

$$\begin{aligned} XJ + YM &= XJ + Y(-a_m J + A) \\ &= (X - a_m Y)\, J + YA. \end{aligned}$$

Taking

$$x = X - a_m Y,\ y = Y; \text{ i.e. } X = x + a_m y,\ Y = y;$$

we obtain transformation formulae for moving from one system of coordinates to the other. The one-one correspondence between the points proves the theorem.

In the new system, M is the point $(0, 1)$ and H is the point $(a_h - a_m, -1)$ as is otherwise clear also from the graph.

3. CHAIN OF SUCCESSORS

Let $A_1 = (a_1, a_2, \ldots, a_k)$.

Then we have

$$\begin{aligned} &[a_2, a_3, \ldots a_k, a_1] - [a_1, a_2, \ldots a_{k-1}, a_k] \\ &\quad = [a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, a_1 - a_k] \qquad \ldots(3.1) \end{aligned}$$

Corresponding to (3.1), we have the cycle

$$A_2 = (\,|\,a_2 - a_1\,|\,, |\,a_3 - a_2\,|\,, \ldots, |\,a_k - a_{k-1}\,|\,, |\,a_1 - a_k\,|\,). \quad \ldots(3.2)$$

We call the cycle A_2 'the successor' of A_1.

Here it should be borne in mind that although A_1 is not a multiple of J; A_2 could be a multiple of J. Note further that A_2 is unique. That is why we have called it *the* successor of A_1.

Moreover, J has no successor nor has any one of its multiples a successor, because when

$$a_1 = a_2 = \dots = a_k$$

A_2 does not belong to C_k, for then each of its elements is zero.

Having called A_2 the successor of A_1, it will be most appropriate to call A_1 'a predecessor' of A_2. We say a predecessor because the predecessor is not unique. For example (0, 9, 14, 25, 2, 17) and (25, 16, 11, 0, 23, 8) are both predecessors of (9, 5, 11, 23, 15, 17).

Denoting by A_{r+1} the successor of A_r, starting from A_1 we can find a chain of successors $A_2, A_3, A_4, \dots, A_t, \dots$.

This chain will end only if we finally arrive at a multiple of J. Otherwise, the chain will be unending.

Remark : The very process of finding the successor of a cycle has an equalizing effect on the elements of the cycle, being in fact similar to the process of finding the g.c.d. of a set of natural numbers. Finally, therefore, not more than two distinct numbers 0 and u say will remain. The cycle will then be a multiple of J or of some cycle whose elements are just 0's and 1's, not all zero of course.

We are now in a position to prove the following.

Theorem 2 — The sum of the elements of the successor of any cycle (when it exists) is an even positive integer.

PROOF : The elements of the matrix in (3.1) and those of the cycle in (3.2) which is the successor of A_1, are the same except for their signs. The sums of the elements of the two must, therefore, be of the same parity. Since the sum of the elements of the matrix in (3.1) is zero, the sum of the elements of A_2 which are all nonnegative and not all zero must be an even positive integer.

The theorem does not imply that a cycle the sum of whose elements is an even positive integer has necessarily a predecessor.

As a corollary of Theorem 2, we have:

For k odd, J has no predecessor. ...(3.3)

(And we already know that it has no successor either).

It would be reasonable here to ask about the nature of the successors of the members of the system generated by a given cycle. The answer is simple to give.

Let A_1 be the given cycle. Take the cycle

$$B_1 = xJ + yA_1.$$

Then, if y is positive, we have

$$B_2 = yA_2.$$

If y is negative, $x \geqslant -ya_h$. Let $x = x' - ya_h$, so that

$$B_1 = x'J - y(a_hJ - A_1).$$

Hence $B_2 = -yA_2$ and $B_r = |\, y \,|\, A_r$, for all r for which it exists.. This means that the successors of the cycles generated by A_1 all lie on the ray emanating from the origin and passing through the point representing the successor of A_1, in the k-dimensional space.

4. Conditions for the Existence of a Predecessor

Any cycle

$$A = (a_1, a_2, \ldots, a_k); \quad A = tJ \text{ permitted};$$

will have a predecessor if and only if there exists a cycle

$$Q = (q_1, q_2, \ldots, q_k)$$

such that

$$\begin{aligned} q_2 - q_1 &= \pm\, a_1 \;, \\ q_3 - q_2 &= \pm\, a_2 \;, \\ &\cdots\cdots\cdots\cdots \\ q_k - q_{k-1} &= \pm\, a_{k-1}, \\ q_1 - q_k &= \pm\, a_k \;. \end{aligned}$$

These relations will hold if and only if for at least one choice of signs on the right of these relations, the sum of the quantities on that side is zero.

That is, the relation

$$0 = \pm\, a_1 \pm a_2 \pm \ldots \pm a_k \qquad \ldots(4.1)$$

holds for at least one choice of signs on the right.

This cannot happen if all the signs chosen are like. Moreover, there is no loss of generality if we choose $+$ for the sign of a_1. Thus, there are $(2^{k-1} - 1)$ relations (4.1), at least one of which must be satisfied by the a's, if A is to have a predecessor. Moreover, if one such relation holds for such a choice, the foregoing relations enable us to express $q_2, q_3, \ldots, q_k$ in terms of q_1 and we can choose q_1 so that not only q_1 but also all other q's will be nonnegative (even positive and as large as we like).

Example — Take the cycle

$$A = (3, 2, 7, 2, 5, 1).$$

Here, we have

$$3 + 2 - 7 - 2 + 5 - 1 = 0$$

as a solution of (4.1).

We, therefore, take

$$\begin{aligned} q_2 - q_1 &= 3, & q_2 &= q_1 + 3; \\ q_3 - q_2 &= 2, & q_3 &= q_1 + 5; \\ q_4 - q_3 &= -7, & q_4 &= q_1 - 2; \\ q_5 - q_4 &= -2, & q_5 &= q_1 - 4; \\ q_6 - q_5 &= 5, & q_6 &= q_1 + 1. \end{aligned}$$

Now any $q_1 \geqslant 4$ will give a predecessor Q of the given cycle. For $q_1 = 4$, we get

$$Q = (4, 7, 9, 2, 0, 5). \qquad \text{...(4.2)}$$

As another solution of (4.1), we have

$$3 - 2 + 7 - 2 - 5 - 1 = 0.$$

Proceeding on the same lines as before, we would get

$$Q^* = (0, 3, 1, 8, 6, 1). \qquad \text{...(4.3)}$$

The reader can find some more predecessors of the given cycle for himself.

Two cycles will be said to be independent when they do not have a common generator i.e. when they are not collateral. We assert that the two Q's obtained above are independent.

PROOF: If they are collateral, let B be their common generator. Then, for some integers x_1, y_1; x_2, y_2; y_1, y_2 not zero, we must have

$$Q = x_1 J + y_1 B, \quad Q^* = x_2 J + y_2 B. \qquad \text{...(4.4)}$$

These relations can hold only if integers u, v, w (u, v not zero) exist such that

$$uQ + vQ^* = wJ.$$

It is easy to show that such u, v, w do not exist and the assertion follows.

5. USE OF MATRICES

The criterion for deciding whether or not a given cycle has a predecessor, provided by (4.1), can best be presented in the form:

$$A_{1\times k} \cdot \begin{bmatrix} 1 \\ J^*_{(k-1)\times 1} \end{bmatrix} = O. \qquad \text{...(5.1)}$$

This result has in this form a very significant interpretation which we present in the following:

Theorem 3 — Every lattice point other than O, with nonnegative coordinates, lying on any of the flats

$$X_1 \pm X_2 \pm \ldots \pm X_k = 0 \qquad \ldots(5.2)$$

represents a cycle which has a predecessor and conversely. The perpendicular at the origin to any of the flats in (5.2) is given by one of the lines OJ^*, $J^* \neq J^{(2k)}$.

Example — For $k = 3$, these flats are the planes :

$$X_1 + X_2 - X_3 = 0,$$

$$X_1 - X_2 + X_3 = 0,$$

$$X_1 - X_2 - X_3 = 0.$$

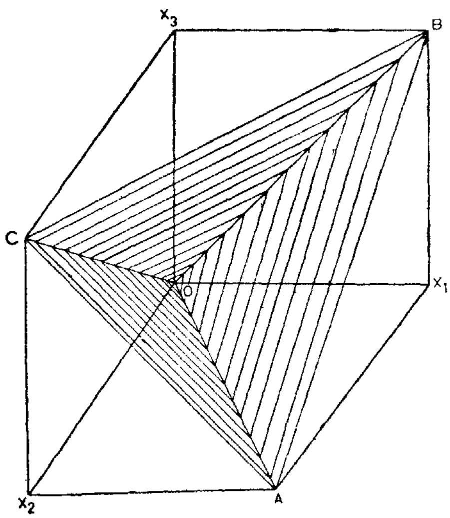

FIG. 2. Planes *BOC, COA* and *AOB* are given by $X_1 + X_2 - X_3 = 0$, $X_1 - X_2 + X_3 = 0$, $X_1 - X_2 - X_3 = 0$.

6. AN ALGORITHM

If in any relation (4.1) satisfied by the elements of any cycle A, the elements with minus signs are transposed to the other side, we get a partition of the elements of A into two sets such that the sum of the elements in one set is the same as the sum of those in the other. This leads to the following:

Theorem 4 — Any given cycle has a predecessor if and only if its elements can be partitioned into two subsets such that the sum of the elements in one subset is equal to the sum of those in the other.

To find if a given cycle has or does not have a predecessor we can, therefore proceed as follows:

Find the sum of the elements of the cycle. If it is odd, the cycle has no predecessor.

If the sum is even $2s$, say, then see if from among the elements of the cycle, you can pick up some (one or more) which add up to s. If you can the cycle has a predecessor, otherwise not.

After we know that the given cycle has a predecessor, we can write out the predecessor as follows:

Let the given cycle be

$$A = (a_1, a_2, \ldots, a_k), \quad (A = tJ \text{ permissible}).$$

In it underline the elements which you picked up and which added up to s.

To get a predecessor Q of A, take

$$q_1 = s,$$

and

$$q_r = q_{r-1} + a_{r-1}, \text{ if } a_{r-1} \text{ is underlined;}$$

$$= q_{r-1} - a_{r-1}, \text{ if } a_{r-1} \text{ is not underlined; } 2 \leqslant r \leqslant k.$$

Remark : Taking $q_1 = s$ ensures that all the q's will be nonnegative and at least one of them positive. After a Q has been found in this manner, it can be replaced by one of its collaterals M or H.

7. An Application

The following is a well-known theorem in graph theory:

Theorem — Any graceful cycle is of order $4t$ or $4t - 1$. The theorem reduces to the following:

The cycle

$$N = (1, 2, 3, \ldots, n)$$

has a predecessor if and only if n is of the form $4t$ or $4t - 1$, $t \geqslant 1$.

Proof : First note that the sum of the elements of N is odd unless n is of the form $4t$ or $4t - 1$. The condition is, therefore, necessary.

If $n = 4t$, the sum of the elements of N is $2t(4t + 1)$.

If from among the elements of N, we pick up the elements

$$t+1, t+2, t+3, \ldots, 3t$$

then since these add up to $t(4t+1)$, it follows that N has a predecessor.

If $n = 4t - 1$, the sum of the elements of N is $2t(4t-1)$.

We now pick up the elements

$$t, t+1, t+2, \ldots, 3t-1$$

which add up to $t(4t-1)$ and we are through.

The conditions are therefore sufficient also.

As a corollary of Theorem 4, we have:

If two cycles have exactly the same elements but not in the same order, then one of these has a predecessor if and only if the other has a predecessor.

We use this to prove that:

Among the cycles with elements $1, 2, 3, \ldots, n$ where $n = 4t$ or $4t-1$, $t \geqslant 1$; there is at least one which has a predecessor no element of which exceeds n.

PROOF : Consider the cycle N, $n = 4t$ or $4t - 1$, $t \geqslant 1$.

Arrange the elements in descending order of magnitude. Now, remove the element $[(n+1)/2]$, where the square bracket denotes the greatest integer function, and put it at the end.

For $n = 4t$, this gives the cycle

$$N^* = (4t, 4t-1, 4t-2, \ldots, 2t+1, 2t-1, 2t-2, \ldots, 2, 1, 2t).$$

Note that the elements

$$4t, 4t-2, \ldots, 2t+2, 2t-1, 2t-3, \ldots, 3, 1 \qquad \ldots(7.1)$$

which appear at the odd places in our arrangement, add up to $t(4t+1)$. Underlining these elements and taking $q_1 = 0$, we get the desired predecessor of N^*, viz.

$$(0, 4t, 1, 4t-1, 2, \ldots, 2t+1, 2t-1, 2t). \qquad \ldots(7.2)$$

It is the existence of such a predecessor that makes N^* graceful. Notice that (7.2) has all the elements from 0 to $4t$ except $3t$.

The same procedure works when $n = 4t - 1$. In this case, the desired predecessor has all the elements from 0 to $4t - 1$ except t.

For $n = 12$, we have the cycle

$$(12, 11, 10, 9, 8, 7, 5, 4, 3, 2, 1, 6)$$

and the predecessor

$$(0, 12, 1, 11, 2, 10, 3, 8, 4, 7, 5, 6).$$

For $n = 11$, the cycle is

$$(11, 10, 9, 8, 7, 5, 4, 3, 2, 1, 6)$$

and the predecessor

$$(0, 11, 1, 10, 2, 9, 4, 8, 5, 7, 6). \qquad \ldots(7.3)$$

When several predecessors for a given cycle are available, it will be reasonable to choose one in which the largest element is the smallest. If there are more than one such, we select that one in which the sum of elements is least.

In the case of $n = 11$, for example, the complement of (7.3) viz.

$$(11, 0, 10, 1, 9, 2, 7, 3, 6, 4, 5) \qquad \ldots(7.4)$$

gives a better result. With this choice, the missing figure is given by $[3n/4]$ whether $n = 4t$ or it is $4t - 1$.

8. Classification of Cycles

If the elements of a cycle are arranged round a circle, as suggested in section 1, and the starting point and the direction in which the elements have been arranged are not indicated, the figure will represent several cycles in general. These cycles will be considered to be equivalent. Thus, for example, the cycles

$$(6, 3, 5, 9), (3, 5, 9, 6), (5, 9, 6, 3), (9, 6, 3, 5);$$

$$(6, 9, 5, 3), (3, 6, 9, 5), (5, 3, 6, 9), (9, 5, 3, 6);$$

are equivalent and their successors are equivalent too.

For any given k, how the chains of successors of the cycles of C_k are going to behave, depends upon the behaviour of cycles whose elements are 0's and 1's, but neither all zero's, nor all one's. Of these, those that have an even number of one's are known to have predecessors. We need, therefore, consider only those that have an odd number of one's. Of these too, it will be enough to study only one from each set of equivalents. Thus, for $k = 6$, we need consider only the five cycles:

$$A = (1, 0, 0, 0, 0, 0);$$

$$B = (1, 1, 1, 0, 0, 0),\ C = (1, 1, 0, 1, 0, 0),\ D = (1, 0, 1, 0, 1, 0);$$

$$E = (1, 1, 1, 1, 1, 0).$$

We deal with each of these in turn, giving the chain of successors.

(i) $A_1 = (1, 0, 0, 0, 0, 0),$

$A_2 = (1, 0, 0, 0, 0, 1),$

$A_3 = (1, 0, 0, 0, 1, 0),$

$A_4 = (1, 0, 0, 1, 1, 1),$

$A_5 = (1, 0, 1, 0, 0, 0) \hookrightarrow A_3.$

We immediately conclude that the chain of successors is unending. The cycles A_1 and A_2 do not reappear. The two cycles A_3 and A_4 recur. We, therefore, say that the cycle A_1 is of the type $(2; \dot{2})$.

Evidently A_2 is a cycle of the type $(1; \dot{2})$ and A_3 and A_4 are both of the type $(0; \dot{2})$.

(ii) Here, we have

$B_1 = (1, 1, 1, 0, 0, 0),$

$B_2 = (0, 0, 1, 0, 0, 1),$

$B_3 = (0, 1, 1, 0, 1, 1),$

$B_4 = (1, 0, 1, 1, 0, 1) \hookrightarrow B_3.$

Hence B_1 is of the type $(2; \dot{1})$, B_2 of the type $(1; \dot{1})$ and B_3 of the type $(0; \dot{1})$.

(iii) We next take

$C_1 = (1, 1, 0, 1, 0, 0),$

$C_2 = (0, 1, 1, 1, 0, 1),$

$C_3 = (1, 0, 0, 1, 1, 1) \hookrightarrow A_4.$

Thus C_1, C_2, C_3 are cycles of the types $(2; \dot{2})$, $(1; \dot{2})$ and $(0; \dot{2})$ respectively.

(iv) Now, we take

$D_1 = (1, 0, 1, 0, 1, 0),$

$D_2 = (1, 1, 1, 1, 1, 1).$

Evidently D_1, D_2 are cycles of the type $(2; 0)$ and $(1; 0)$ respectively.

(v) Finally, we have

$E_1 = (1, 1, 1, 1, 1, 0),$

$E_2 = (0, 0, 0, 0, 1, 1) \hookrightarrow A_2.$

Since A_2 is a cycle of the type $(1; \dot{2})$, so is E_2 and E_1 is a cycle of the type $(2; \dot{2})$.

The above information is enough for us now to determine the type of any cycle of order 6, with the minimum effort.

Take, for example

$$F_1 = (3, 4, 7, 2, 1, 4).$$

Then

$$F_2 = (1, 3, 5, 1, 3, 1),$$

$$F_3 = (2, 2, 4, 2, 2, 0) = 2(1, 1, 2, 1, 1, 0),$$

$$F_4 = 2(0, 1, 1, 0, 1, 1) = 2B_3.$$

Since B_3 is of the type $(0; \dot{1})$, we conclude that F_1 is of the type $(3; \dot{1})$.

For any cycle of type $(r; \dot{p})$, we shall say that the cycle is of rank r and periodicity p.

It will be seen that for any $k > 2$, the periodicity is bounded.

The following theorems are easy to prove.

Theorem 5 — Collateral cycles are all of the same type as their generator.

Theorem 6 — No cycle of an odd order $k \geqslant 3$, can be of the terminating type.

9. Terminating Zero-one Cycles

As in the preceding section, let

$$A = (a_1, a_2, \ldots, a_k)$$

be a cycle which has $a_t = 1$ for an odd number of values of $t \leqslant k$ and $a_t = 0$ for the rest of them. (This means that the cycle has no predecessor). Evidently if A is a terminating cycle, then k must be even. Let $k = 2^j m$, with $j \geqslant 1$ and m odd and $\geqslant 1$. Since for a zero-one cycle such as our A, the process of finding successors is addition of successive members modulo 2, with this modulus throughout this section, we have

$$A_1 \equiv (a_1 + a_2, \ldots, a_t + a_{t+1}, \ldots, a_k + a_1);$$

$$A_2 \equiv (a_1 + 2a_2 + a_3, \ldots, a_t + 2a_{t+1} + a_{t+2}, \ldots, a_k + 2a_1 + a_2);$$

and so on.

It will be readily seen that the tth element of A_N is

$$\equiv \binom{N}{0} a_t + \binom{N}{1} a_{t+1} + \ldots + \binom{N}{N} a_{t+N},$$

where the subscripts of a are reducible modulo k.

Now for $N = 2^j - 1$, it is well known that $\binom{N}{r}$ is odd for all r, $0 \leqslant r \leqslant N$. Hence the tth element of A_{2^j-1}

$$\equiv a_t + a_{t+1} + \ldots + a_{t+2^j-1},\ 1 \leqslant t \leqslant k.$$

Hence A is terminating if and only if

$$a_t + a_{t+1} + \ldots + a_{t+2^j-1} \text{ is odd for each } t \leqslant k.$$

This implies that for each $t \leqslant 2^j$,

$$a_t = a_{t+2^j} = a_{t+2 \cdot 2^j} = \ldots = a_{t+(m-1)2^j}.$$

Hence A is terminating if it is of the form

$$A = (\overline{a_1, a_2, \ldots, a_{2^j}}, \text{ juxtaposed } m \text{ times})$$

with

$$a_1 + a_2 + \ldots + \alpha_{2^j} \text{ odd}.$$

For $m = 1$, this condition is already satisfied by any of our A's. We have thus proved the following.

Theorem 7 — For k even, terminating zero-one cycles always exist and they have a particular pattern when $k \neq 2^j$.

For $k = 2^j$ every cycle of order k whether or not of the zero-one type is terminating.

10. Existence of the Cycles of all Ranks

Theorem 8 — Among the cycles generated by M, there is at least one which has a predecessor.

Proof : We first assert that either the elements of M or those of H can be partitioned into two unequal subsets such that the sum of elements in the shorter subset is not less than the sum of the elements in the longer subset.

Let the elements of M be arranged in non-descending order of magnitude.

For $k = 2t + 1$, $t \geqslant 1$, the sum of the last t elements on the right is certainly not less than the sum of the $(t + 1)$ elements on the left because one at least of these elements is zero. This proves our assertion for odd $k \geqslant 3$.

For $k = 2t$, $t \geqslant 2$, if the sum of the last $(t - 1)$ elements on the right is not $\geqslant$ the sum of the $(t + 1)$ elements on the left, then shift the extreme left hand

Theorem 7 and its proof is essentially due to S. Srinivasan, Tata Institute of Fundamental Research, Bombay (India).

zero to the extreme right. Now the sum of the t elements on the right is less than the sum of the t elements on the left. The sum of the complements of the t elements on the right is, therefore, greater than the sum of the complements of the t elements now on the left. Of the complements of the t elements on the right the complement of the largest element of M is necessarily zero. Shifting this zero to the extreme left, we conclude that the elements of H can be partitioned into two subsets one with $(t - 1)$ elements having a larger sum than the sum of the $(t + 1)$ elements in the other.

Our assertion is thus completely proved.

Before proceeding further with the proof, we will give an example.

Example — Let

$$M = (6, 3, 0, 4, 5, 5).$$

Then we have

First step :	0, 3, 4, 5 | 5, 6	
Second step :	3, 4, 5 | 5, 6, 0	$0 + 5 + 6 < 3 + 4 + 5$
Third step :	3, 2, 1 | 1, 0, 6	
Fourth step :	0, 3, 2, 1 | 1, 6	$1 + 6 > 3 + 2 + 1 + 0$

Thus the elements of $H = (0, 3, 6, 2, 1, 1)$ have been partitioned in the desired manner.

Now assume that neither M nor H has a predecessor, for if any one of them has a predecessor we have nothing to prove.

Suppose the elements of M or H have been partitioned in the desired manner and the shorter subset has r_1 elements whose sum is s_1, and longer subset has r_2 elements which add up to s_2.

Then, the cycle $xJ + yM$ or the cycle $xJ + yH$ will have a predecessor, if we can find integers x and y with $y \neq 0$, such that

$$xr_1 + ys_1 = xr_2 + ys_2;$$

i.e. $$(r_2 - r_1)\, x = (s_1 - s_2)\, y. \tag{10.1}$$

Since by our assumption,

$$r_2 > r_1, \text{ and } s_1 > s_2;$$

(10.1) has definitely a solution in positive integers x, y. (We of course prefer the smallest). This proves the theorem. The following example will show how a solution of (10.1) in which x, y will be as small as possible can be obtained. As a general rule, it is best to first try the partitions of M or H in which the shorter subset has t elements and the longer one $t + 1$ when $k = 2t + 1$; and those in which the shorter subset has $(t - 1)$ elements and the longer one $(t + 1)$ when $k = 2t$.

Example — Consider the system generated by

$$M = (1, 0, 8, 5, 4, 6, 18).$$

The results are best presented in a tabular form :

	Partition	$r_2 - r_1$	$s_1 - s_2$	x	y
(1)	18, 4, 0 | 1, 5, 6, 8	1	2	2	1
(2)	18, 5, 0 | 1, 4, 6, 8	1	4	4	1
(3)	18, 6, 0 | 1, 4, 5, 8	1	6	6	1
(4)	18, 8, 0 | 1, 4, 5, 6	1	10	10	1
(5)	18, 4, 1 | 0, 5, 6, 8	1	4	4	1
(6)	18, 5, 1 | 0, 4, 6, 8	1	6	6	1
(7)	18, 6, 1 | 0, 4, 5, 8	1	8	8	1
(8)	18, 8, 1 | 0, 4, 5, 6	1	12	12	1

Evidently no good is going to come out of further tabulation with 3 elements in the shorter subset and 4 in the longer one. We now try some more with fewer elements in the shorter subset

(9)	18, 4 | 0, 1, 5, 6, 8	3	2	2	3
(10)	18, 5 | 0, 1, 4, 6, 8	3	4	4	3
(11)	18, 6 | 0, 1, 4, 5, 8	3	6	2	1
(12)	18, 8 | 0, 1, 4, 5, 6	3	10	10	3

Before taking a final decision, let us have a look at H too because, it generates the same system as M.

We have $H = (17, \underline{18}, \underline{10}, 13, \underline{14}, 12, 0)$.

It turns out that this has itself a predecessor.

So, the choice lies between the cycles given by H and (1) or (11). Now (1) and (11) both give the same cycle viz.

$$(3, 2, 10, 7, 6, 8, 20).$$

Our choice must, therefore, fall on H.

Theorem 9 — For each $k > 2$, cycles of all ranks exist and can be constructed.

PROOF : Let A be any cycle of order k with at least two unequal elements. Suppose it is of the type $(r; \dot{p})$.

Then, using the method described earlier, we can find among the cycles generated by A, one which has a predecessor. This predecessor is of the type $(r + 1; \dot{p})$.

Moreover, the successors of A, provide cycles of types

$(t; \dot{p})$, where $0 \leqslant t \leqslant r-1$ if p is not zero, and $1 \leqslant t \leqslant r-1$ otherwise. This proves the theorem.

Note that for $k = 2$, all cycles are of rank 1 or 2 and they are terminating.

11. The Cases $k = 3, 4$

The two cases are of special interest. For $k = 3$ all cycles are of the unending type with periodicity 1 (we exclude, the cycle J of course). For $k = 4$ the cycles are all terminating.

Theorem 10 — The cycle

$(0, n, n-1)$ is of the type $(n; \dot{1})$.

Proof : It is easy to verify that the cycle $(0, 1, 0)$ is of the type $(1; \dot{1})$. Assume that the theorem is true for some $n > 1$. Then, among the cycles generated by $(0, n, n-1)$, the cycle

$$(1, n+1, n) \hookrightarrow (n+1, 1, n)$$

has a predecessor which is given by $(0, n+1, n)$. This is of the type $(n+1; \dot{1})$ and the theorem follows by induction.

The case $k = 4$.

Here, we give a table in which we record one cycle of each rank $\leqslant 15$ and give a formula for constructing a cycle of any higher rank. Where the cycle has no predecessor, we record in the same row a cycle which has a predecessor. This is then recorded in the next row.

Rank	*Cycle*	
1	(1, 1, 1, 1)	
2	(0, 1, 0, 1)	
3	(0, 0, 1, 1)	
4	(0, 0, 0, 1)	(1, 1, 1, 3)
5	(0, 1, 2, 3)	
6	(0, 0, 1, 3)	(1, 1, 2, 4)
7	(0, 1, 2, 4)	(1, 3, 5, 9)
8	(0, 1, 4, 9)	(2, 3, 6, 11)
9	(0, 2, 5, 11)	(2, 4, 7, 13)
10	(0, 2, 6, 13)	(5, 9, 17, 31)
11	(0, 5, 14, 31)	(6, 11, 20, 37)
12	(0, 6, 17, 37)	(7, 13, 24, 44)
13	(0, 7, 20, 44)	(17, 31, 57, 105)
14	(0, 17, 48, 105)	(20, 37, 68, 125)
15	(0, 20, 57, 125)	....

Let $R(r)$ denote in the above table, the cycle for rank r. Then for $r \geqslant 8$, we have the relation:

$$R(r) = R(r-3) + R(r-2), \quad \text{if } r \equiv 1 \pmod 3;$$
$$= R(r-3) + 2R(r-2), \text{ if } r \equiv 0, 2 \pmod 3.$$

Here addition means matrix addition.

Thus

$$R(16) = R(13) + R(14) = (0, 24, 68, 149).$$

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