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SOME THEOREMS ON INFINITESIMAL CONFORMAL TRANSFORMATION IN PROJECTIVE SYMMETRIC FINSLER SPACE

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In this paper we have studied infinitesimal conformal transformation in projective symmetric Finsler space. Some theorems have been investigated.

1. INTRODUCTION

We consider an n-dimensional Finsler space F_n (Rund 1959) having fundamental metric tensor $g_{ij}(x, \dot{x})$ which is given by

$$g_{ij}(x, \dot{x}) \overset{def}{=} \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}),\ \dot{\partial}_i \equiv \partial/\partial\dot{x}^i \qquad ...(1.1a)$$

and

$$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } k = i \\ 0 & k \neq i. \end{cases} \qquad ...(1.1b)$$

The metric tensor $g_{ij}(x, \dot{x})$ satisfies the following identities:

$$\dot{\partial}_r g_{kh} = 2C_{rkh} \qquad ...(1.2a)$$

where

$$C_{rkh}\dot{x}^r = 0. \qquad ...(1.2b)$$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt \qquad ...(1.3)$$

where $v^i(x)$ is a vector field and dt is an infinitesimal constant. The covariant derivative of a tensor field $T^i_j(x, \dot{x})$ introduced in (Rund 1959)

$$T^i_{j(k)} = \partial_k T^i_j + \dot{\partial}_h T^i_j\, G^h_k + T^h_j\, G^i_{hk} - T^i_h\, G^h_{jk} \qquad ...(1.4)$$

where the function $G^i(x, \dot{x})$ is positively homogeneous of degree two in the $\dot{x}^i$. This function satisfies the following identities:

$$(a)\quad G^i_{hkr}\,\dot{x}^r = 0,\quad (b)\quad G^i_{hk}\,\dot{x}^k = G^i_h,\quad (c)\quad \dot{\partial}_h\dot{\partial}_k G^i = G^i_{hk}. \qquad ...(1.5)$$

Using the covariant derivative and the infinitesimal point change (1.2) the Lie-derivative of a tensor field and connection parameter are given by (Yano 1957).

$$\mathcal{L}_v T^i_j\,(x, \dot{x}) = T^i_{j(r)}\, v^r - (\dot{\partial}_s T^i_j)\, V^{(s)}_r\, \dot{x}^r - T^r_j\, V^i_{(r)} + T^i_r\, V^r_{(j)} \qquad \text{...(1.6)}$$

and

$$\mathcal{L}_v G^i_{lm} = V^i_{(l)(m)} + H^i_{lmr}\, V^r + (\dot{\partial}_r G^i_{lm})\, V^r_{(s)}\, \dot{x}^s \qquad \text{...(1.7)}$$

where

$$H^i_{lmr} = 2\partial_{[r} G^i_{m]l} + 2G^j_{l[m} G^i_{r]j} + 2G^i_{jl[r}\, \dot{\partial}_{m]}\, G^j. \qquad \text{...(1.8)}$$

The curvature tensor H^i_{jkh} satisfies the following identities and contractions

$$\left.\begin{array}{lll} \text{(a)} \quad H^i_{jkh} = -H^i_{jhk}, & \text{(b)} \quad H^j_k\, \dot{x}^k = 0 & \\ \text{(c)} \quad H_{jk}\dot{x}^j = H_k, & \text{(d)} \quad H_{ij} = H^l_{ijl} & \\ \text{(e)} \quad 2H_{[jk]} = H^l_{lkj}, & \text{(f)} \quad H_j = H^l_{jl}, & \text{(g)} \quad H = H^i_i/(n-1). \end{array}\right\} \qquad \text{...(1.9)}$$

Between the operators $\mathcal{L}_v$, $\dot{\partial}$ and (k), we can obtain the following commutation formulae

$$\mathcal{L}_v(\dot{\partial}_k T^i_j) - \dot{\partial}_k(\mathcal{L}_v T^i_j) = 0 \qquad \text{...(1.10)}$$

$$\mathcal{L}_v(T^i_{j(k)}) - (\mathcal{L}_v T^i_j)_{(k)} = T^h_j\, \mathcal{L}_v G^i_{kh} - T^i_h\, \mathcal{L}_v G^h_{kj} - (\dot{\partial}_h T^i_j)\, \mathcal{L}_v G^h_{ks}\, \dot{x}^s \qquad \text{...(1.11)}$$

and

$$(\mathcal{L}_v G^i_{jh})_{(k)} - (\mathcal{L}_v G^i_{kh})_{(j)} = \mathcal{L}_v H^i_{hjk} + (\mathcal{L}_v G^r_{kl})\, G^i_{rjh}\, \dot{x}^l - (\mathcal{L}_v G^r_{jl})\, \dot{x}^l G^i_{rhk}. \qquad \text{...(1.12)}$$

The projective deviation tensor field $W^i_j\,(x, \dot{x})$ is given by

$$W^i_j\,(x, \dot{x}) = H^i_j - H\delta^i_j - \frac{\dot{x}^i}{(n+1)}\,(\dot{\partial}_r H^r_j - \dot{\partial}_j H) \qquad \text{...(1.13)}$$

and satisfies the following identities

$$\left.\begin{array}{ll} \text{(a)} \quad W^i_l\, \dot{x}^l = 0, & \text{(b)} \quad \dot{\partial}_l W^i_j\, \dot{x}^l = 2W^i_j \\ \text{(c)} \quad W^i_i = 0, & \text{(d)} \quad \dot{\partial}_i W^i_j = 0. \end{array}\right\} \qquad \text{...(1.14)}$$

We have the following commutation formula

$$\dot{\partial}_j(T^i_{k(r)}) - (\dot{\partial}_j T^i_k)_{(r)} = T^m_k G^i_{jmr} - T^i_m G^m_{jkr} \cdot \qquad ...(1.15)$$

2. Infinitesimal Conformal Transformation

The necessary and sufficient condition that an infinitesimal point transformation (1.3) be an infinitesimal conformal transformation, following equation (Izumi 1977)

$$\mathcal{L}_v G^i_{jk} = \delta^i_j \epsilon_k + \delta^i_k \epsilon_j - \epsilon^i g_{jk} \qquad ...(2.1)$$

must hold. Here $\epsilon_{k(x)}$ and $\epsilon^i(x)$ are covariant and contravariant vector fields.

With the help of commutation formula (1.12), the Lie-derivative of H^i_{hjk} is given by

$$\mathcal{L}H^i_{hjk} = (\mathcal{L}G^i_{jk})_{(k)} - (\underset{v}{\mathcal{L}}G^i_{kh})_{(j)} - \mathcal{L}_v(G^r_{kl})\, G^i_{rjh}\, \dot{x}^l + (\mathcal{L}_v G^r_{jl})\, \dot{x}^l G^i_{rhk}. \qquad ...(2.2)$$

Using eqns. (1.5) and (2.1) in (2.2), we get

$$\begin{aligned}\mathcal{L}_v H^i_{hjk} = {} & \delta^i_j \epsilon_{h(k)} + \delta^i_h \epsilon_{j(k)} - g_{jh(k)}\epsilon^i - g_{jh}\epsilon^i_{(k)} - \delta^i_k \epsilon_{h(j)} - \delta^i_h \epsilon_{k(j)} \\ & + g_{kh(j)}\epsilon^i + g_{kh}\epsilon^i_{(j)} - \delta^r_l \epsilon_k G^i_{rjh}\, \dot{x}^l + g_{kr}\epsilon^r G^i_{rjh}\, \dot{x}^l \\ & + \delta^r_l \epsilon_j G^i_{rhk}\, \dot{x}^l - g_{jl}\epsilon^r G^i_{rhk}\, \dot{x}^l. \end{aligned} \qquad ...(2.3)$$

Transvecting (2.3) by $\dot{x}^h \dot{x}^j$ and noting the homogeneity property of H^i_{hjk}, we get

$$\mathcal{L}_v H^i_k = -\, \delta^i_h \epsilon_{k(j)}\dot{x}^h \dot{x}^j + g_{kh(j)}\epsilon^i \dot{x}^h \dot{x}^j + g_{kh}\epsilon^i_{(j)}\, \dot{x}^h \dot{x}^j. \qquad ...(2.4)$$

Contracting the above equation with respect to indices i and k and using eqn. (1.9a), we obtain

$$\mathcal{L}_v H = 0. \qquad ...(2.5)$$

With the help of eqn. (2.4) and (2.5), we get

$$\mathcal{L}_v H^i_k - \mathcal{L}_v H \delta^i_k = -\epsilon_{k(j)}\dot{x}^i \dot{x}^j + g_{kh(j)}\epsilon^i \dot{x}^h \dot{x}^j + g_{kh}\epsilon^i_{(j)}\, \dot{x}^h \dot{x}^j. \qquad ...(2.6)$$

Differentiating (2.6) partially with respect to $\dot{x}^r$ and contracting the resulting equation with respect to i and r, we get

$$\mathcal{L}_v(\dot{\partial}_r H_k) - \mathcal{L}_v(\dot{\partial}_r H) = -(\dot{\partial}_r \epsilon_k)_{(j)}\, \dot{x}^r \dot{x}^j - \epsilon_{k(j)}\dot{x}^j$$
$$- \epsilon_{k(r)}\dot{x}^r + g_{kh(j)}(\dot{\partial}_r \epsilon^r)\, \dot{x}^h \dot{x}^r + g_{kr(j)}\epsilon^r \dot{x}^j$$
$$+ g_{kh(r)}\epsilon^r \dot{x}^h + g_{kh}(\dot{\partial}_r \epsilon^r)_{(j)}\, \dot{x}^h \dot{x}^j$$
$$+ g_{kr}\,\epsilon^r_{(j)}\, \dot{x}^j + g_{kh}\,\epsilon^r_{(j)}\, \dot{x}^h. \qquad \ldots(2.7)$$

The Lie-derivative of the projective deviation tensor field $W^i_j(x, \dot{x})$ is given by

$$\mathcal{L}_v W^i_k(x, \dot{x}) = [g_{hh(j)}\epsilon^i \dot{x}^h \dot{x}^j + g_{kh}\,\epsilon^i_{(j)}\, \dot{x}^h \dot{x}^j - \epsilon_{k(j)}\dot{x}^i \dot{x}^j]$$
$$- \frac{\dot{x}^i}{n+1}\, [(-\, \dot{\partial}_r \epsilon_k)_{(j)}\, \dot{x}^r \dot{x}^j - \epsilon_{k(j)}\dot{x}^j - \epsilon_{k(r)}\dot{x}^r$$
$$+ g_{kh(j)}(\dot{\partial}_r \epsilon^r)\, \dot{x}^h \dot{x}^j + g_{kr(j)}\epsilon^r \dot{x}^j + g_{kh(r)}\epsilon^r \dot{x}^h$$
$$+ g_{kh}(\dot{\partial}_r \epsilon^r)_{(j)}\, \dot{x}^h \dot{x}^j + g_{kr}\,\epsilon^r_{(j)}\, \dot{x}^j + g_{kh}\,\epsilon^r_{(r)}\, \dot{x}^h]. \qquad \ldots(2.8)$$

3. Some Theorems

Applying the commutation formula (1.11) to the projective deviation tensor field $W^i_j(x, \dot{x})$, we get

$$\mathcal{L}_v(W^i_{j(r)}) - (\mathcal{L}_v W^i_j)_{(r)} = W^h_j\, \mathcal{L}_v G^i_{rh} - W^i_h\, \mathcal{L}_v G^h_{jr} - (\dot{\partial}_r W^i_j)\, \mathcal{L}_v G^h_{rs}\, \dot{x}^s. \qquad \ldots(3.1)$$

From eqns. (2.1) and (3.1), we get

$$\mathcal{L}_v(W^i_{j(r)}) - (\mathcal{L}_v W^i_j)_{(r)} = W^h_j\, (\delta^i_r\, \epsilon_h + \delta^i_h\, \epsilon_r - \epsilon^i g_{rh})$$
$$- W^i_h\, (\delta^h_j\, \epsilon_r + \delta^h_r\, \epsilon_j - \epsilon^h g_{jr})$$
$$- (\dot{\partial}_h W^i_j)\, (\delta^h_s\, \epsilon_r - \epsilon^h g_{rs})\, \dot{x}^s. \qquad \ldots(3.2)$$

Contracting (3.2) with respect to indices i and r and noting $(1.14d)$, we get

$$\mathcal{L}_v(W^i_{j(i)}) - (\mathcal{L}_v W^i_j)_{(i)} = (n-1)\, W^i_j\, \epsilon_i + W^i_h\, \epsilon^h g_{ji} + (\dot{\partial}_h W^i_j)\, \epsilon^h g_{is}\dot{x}^s. \qquad \ldots(3.3)$$

Transvecting (3.2) by $\dot{x}^r$ and using relation (1.14), we get

$$\{\mathcal{L}_v(W^i_{j(r)}) - (\mathcal{L}_v W^i_j)_{(r)}\}\, \dot{x}^r = W^h_j\, \epsilon_h \dot{x}^i - W^h_j\, \epsilon^i g_{rh}\dot{x}^r + W^i_h\, \epsilon^h g_{jr}\dot{x}^r$$
$$+ (\dot{\partial}_r W^i_j)\, \epsilon^h g_{rs}\dot{x}^r \dot{x}^s. \qquad \ldots(3.4)$$

Let us now assume that the infinitesimal projective transformation leaves invariant the vanishing of the covariant derivative of W^i_j i.e.

$$\mathcal{L}_v W^i_{j(r)} = 0. \qquad ...(3.5)$$

In view of (3.5), (3.3) and (3.4) take the following forms

$$(\mathcal{L}_v W^i_j)_{(i)} = (1-n)\, W^h_j \epsilon_h - W^i_h \epsilon^h g_{ji} - (\dot{\partial}_h W^i_j)\, \epsilon^h g_{is} \dot{x}^s \qquad ...(3.6)$$

and

$$(\mathcal{L}_v W^i_j)_{(r)}\, \dot{x}^r = W^h_j \epsilon^i g_{rh} \dot{x}^r - W^h_j \epsilon_h \dot{x}^i - W^i_h \epsilon^h g_{jr} \dot{x}^r - (\dot{\partial}_r W^i_j) \epsilon^h g_{rs} \dot{x}^r \dot{x}^s \qquad ...(3.7)$$

respectively.

Eliminating $W^h_j \epsilon_h$ from eqns. (3.6) and (3.7), we get

$$(\mathcal{L}_v W^i_j)_{(i)}\, \dot{x}^i + (1-n)\,(\mathcal{L}_v W^i_j)_{(r)}\, \dot{x}^r$$
$$= (1-n)\, W^h_j \epsilon^i g_{rh} \dot{x}^r - W^i_h \epsilon^h g_{ji} - (\dot{\partial}_h W^i_j)\, \epsilon^h g_{is} \dot{x}^s$$
$$- (1-n)\, W^i_h \epsilon^h g_{jr} \dot{x}^r - (1-n)\,(\dot{\partial}_r W^i_j)\, \epsilon^h g_{rs} \dot{x}^r \dot{x}^s. \qquad ...(3.8)$$

If an F_n admits an affine motion, then the equation

$$\mathcal{L}_v G^i_{jk} = 0 \qquad ...(3.9)$$

must hold.

Therefore, with the help of eqns. (2.1) and (3.9) it is clear that the vectors ϵ^i and ϵ_i must separately vanish. Thus, we have the following.

Theorem 3.1 — If an F_n admits a non-affine infinitesimal conformal transformation such that the Berwald's covariant derivative of W^i_j remains invariant, then eqn. (3.8) holds.

Theorem 3.2 — If an F_n admits an affine infinitesimal conformal transformation characterized by (2.1) such that the covariant derivative of W^i_j remains invariant, then

$$\{(\mathcal{L}_v W^i_j)_{(i)}\, \dot{x}^i + (1-n)\,(\mathcal{L}_v W^i_j)_{(r)}\, \dot{x}^r\} = 0 \qquad ...(3.10)$$

in particular, if the Finsler space becomes projective symmetric Finsler space i.e. $W^i_{j(r)} = 0$ holds, then eqn. (3.5) always holds. Thus, we have the following.

Theorem 3.3 — If an F_n admits a non-affine infinitesimal conformal transformation, then eqn. (3.8) necessarily holds.

Theorem 3.4 — If an F_n admits an affine infinitesimal conformal transformation defined by (2.1), then eqn. (3.10) necessarily holds.

REFERENCES

Izumi, H. (1977). Conformal transformation of Finsler spaces. *Tensor* (*N.S.*), **31**, 33-41.,

Matsumoto, M. (1968). On Riemannian spaces with recurrent projective curvature. *Tensor*, **19**, 11-18.

Rund, H. (1959). The Differential Geometry of Finsler Spaces. Springer-Verlag, Berlin.

Yano, K. (1957). The Theory of Lie-derivative and Its Applications. North-Holland Publishing Co., Amsterdam.