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SOME INEQUALITIES FOR NON-HOMOGENEOUS QUADRATIC FORMS

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For $0 \leqslant \mu < 1$, functions $f(\mu)$ are obtained such that for any real indefinite quadratic form $Q(x, y, z)$ of type (1, 2) and determinant D and real x_0, y_0, z_0, the inequality

$$\mu(f(\mu) D)^{1/3} < Q(x + x_0, y + y_0, z + z_0) < (f(\mu) D)^{1/3}$$

has a solution in integers x, y, z.

This result is used to prove that for any real quaternary from $Q(x, y, z, t)$ of type (1, 3) and determinant D and real numbers x_0, y_0, z_0, t_0, the inequality

$$0 < Q(x + x_0, y + y_0, z + z_0, t + t_0) < \left(\frac{128}{25}(2\sqrt{7} - 1) \mid D \mid\right)^{1/4}$$

has a solution in integers x, y, z, and t.

1. INTRODUCTION

Let $Q(x_1, x_2, ..., x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n - r)$, $0 < r < n$ and determinant $D \neq 0$. Blaney (1948) proved that there exist constants Γ such that for any such quadratic form Q and any real numbers $c_1, c_2, ..., c_n$ we can find integers $x_1, x_2, ..., x_n$ satisfying

$$0 < Q(x_1 + c_1, x_2 + c_2, ..., x_n + c_n) \leqslant (\Gamma \mid D \mid)^{1/n}. \qquad ...(1.1)$$

Let $\Gamma_{r,n-r}$ denote the greatest lower bound of all such constants Γ. Davenport and Heilbronn (1947) showed that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ and $\Gamma_{1,2} = 8$ were proved by Barnes (1961) and Dumir (1967) respectively. Dumir (1968a, b) has also shown that $\Gamma_{3,1} = 16/3$ and $\Gamma_{2,2} = 16$. In this paper we shall prove that

$$16 \leqslant \Gamma_{1,3} \leqslant 128(2\sqrt{7} - 1)/25 = 21.972 \ldots .$$

The motivation for these estimates is to prove in another paper* that the symmetrical non-homogeneous minimum for quadratic forms in five variables of the type (4, 1) or (1, 4) is $\frac{1}{2}$; thus proving in this case a conjecture of Watson (1962). We shall also use this bound on $\Gamma_{1,3}$ to prove $\Gamma_{4,1} = 8$ in another paper. Here we prove:

Theorem 1 — Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form of the type (1, 3) and determinant $D(< 0)$. Then given any real numbers x_0, y_0, z_0, t_0 we can find integers x, y, z, t such that

*See pages 75–91.

$$0 < Q(x + x_0, y + y_0, z + z_0, t + t_0) < (K \mid D \mid)^{1/4} \qquad \ldots(1.2)$$

where $K = 128(2\sqrt{7} - 1)/25$.

Theorem 2 — Let $Q(x, y, z, t) = -x^2 - xz - y^2 - z^2 - yz + 2zt$. Then the inequality

$$0 < Q(x, y, z, t + \tfrac{1}{2}) < (16 \mid D \mid)^{1/4}$$

is not solvable in integers x, y, z and t.

In order to prove Theorem 1, we prove an asymmetric inequality for indefinite ternary quadratic forms. This result is analogous to that of Blaney (1950). He has proved that if $Q(x, y)$ is an indefinite binary quadratic form of discriminant $\triangle^2 > 0$ and x_0, y_0 are any real numbers then there exist integers x, y satisfying

$$\frac{\nu^2 \triangle}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}} < Q(x + x_0, y + y_0) \leqslant \frac{\triangle}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}}$$

where $0 \leqslant \nu < 1$ is a real number. Here we prove:

Theorem 3 — Let $Q(x, y, z)$ be an indefinite ternary quadratic form of type (1, 2) and determinant $D(> 0)$. Let $0 \leqslant \mu < 1$. Then given any real numbers x_0, y_0, z_0 there exist integers x, y, z satisfying

$$\mu(f(\mu)\, D)^{1/3} < Q(x + x_0, y + y_0, z + z_0) < (f(\mu)\, D)^{1/3} \qquad \ldots(1.4)$$

where f is any function satisfying

$$f(\mu) \geqslant \max\,(f_1(\mu), f_2(\mu)) \quad \text{for} \quad 0 \leqslant \mu < 1 \qquad \ldots(1.5)$$

and where

$$f_1(\mu) = \frac{32}{3(1-\mu)^3}$$

$$f_2(\mu) = \frac{4^3}{5^3} \cdot \frac{1}{(1-\mu)^4} \cdot \frac{(\sqrt{16 - 4\mu} + \sqrt{11\mu + 1})^3}{(\sqrt{16 - 4\mu} + 3\sqrt{11\mu + 1})}.$$

2. Some Lemmas

In the course of the proofs we shall use the following lemmas.

Lemma 1 — Let $Q(x_1, \ldots, x_n)$ be an indefinite quadratic form in $n(\geqslant 3)$ variables with real coefficients. Suppose that Q is non-singular and takes arbitrarily small non-zero values for integers $x_1, \ldots, x_n$. Let $c_1, \ldots, c_n, \alpha, \delta$ be real numbers, δ being positive. Then we can find integers $x_1, x_2, \ldots, x_n$ satisfying

$$\mid Q(x_1 + c_1, \ldots, x_n + c_n) - \alpha \mid < \delta. \qquad \ldots(2.1)$$

This is Theorem 1 of Watson (1960).

Lemma 2 — Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form of the type (3, 1) and determinant $D(< 0)$. Then there exist integers x_1, y_1, z_1, t_1 such that

$$0 < Q(x_1, y_1, z_1, t_1) \leqslant \left(\frac{16}{3} \mid D \mid\right)^{1/4} \qquad \ldots(2.2)$$

This is Theorem 2 of Oppenheim (1953).

Lemma 3 — Let $Q(x, y, z)$ be an indefinite ternary quadratic form of the type (2, 1) and determinant $D(< 0)$. Then there exist integers u, v, w such that

$$0 < Q(u, v, w) \leqslant (\tfrac{4}{3} \mid D \mid)^{1/3} \qquad \ldots(2.3)$$

unless $Q(x, y, z)$ is equivalent to a positive multiple of one of the following eight forms

$$Q_1 = x^2 + yz \qquad Q_5 = 2x^2 + 3yz$$

$$Q_2 = 3x^2 + 4yz \qquad Q_6 = 4x^2 + z(4y - z)$$

$$Q_3 = 12x^2 + z(8y - z + 4x) \qquad Q_7 = 9x^2 + z(6y - z + 3x)$$

$$Q_4 = 2x^2 + yz \qquad Q_8 = 16x^2 + z(8y - z).$$

Equality in (2.3) is necessary if and only if Q is equivalent to a positive multiple of one of the following forms

$$Q_9 = 2(x + \tfrac{1}{2} y)^2 + \tfrac{3}{2} y^2 - \tfrac{1}{4} z^2)$$

$$Q_{10} = 3x^2 + yz.$$

This is a Theorem of Watson (1968).

Lemma 4 — Let α, β, d be real numbers with $\beta^2 > 1/4$ and $d \geqslant 1$. Then for any real number x_0 we can find $x \equiv x_0 \pmod 1$ such that

$$0 < -(x + \alpha)^2 + \beta^2 \leqslant d \qquad \ldots(2.4)$$

provided that

$$\beta^2 \begin{cases} \leqslant \left(\dfrac{d+1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ < \left(\dfrac{[d]}{2}\right)^2 + d & \text{if } d \text{ is not an integer.} \end{cases} \qquad \ldots(2.5)$$

Further strict inequality in (2.5) implies strict inequality in (2.4). This is Lemma 2 of Dumir (1967).

Lemma 5 — Let $0 \leqslant \nu < 1$ be a real number and $\varphi(y, z)$ be an indefinite binary quadratic form of discriminant $\Delta^2 > 0$. Then for any real numbers y_0, z_0 there exist integers y and z such that

$$\frac{\nu^2 \Delta}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}} < \varphi(y + y_0, z + z_0) \leqslant \frac{\Delta}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}}. \quad \ldots(2.6)$$

Equality occurs in (2.6) if and only if

$$\nu = 0 \text{ and } \varphi(y, z) \sim \rho yz, (y_0, z_0) = (0, 0) \pmod 1$$

or

$$\varphi(y, z) \sim \rho(y^2 - z^2), (y_0, z_0) = (\tfrac{1}{2}, \tfrac{1}{2}) \pmod 1$$

$$\nu = \tfrac{1}{3}, \quad \varphi(y, z) \sim \rho(3y^2 - z^2), (y_0, z_0) = (\tfrac{1}{2}, \tfrac{1}{2}) \pmod 1$$

$$\nu = \tfrac{1}{2}, \quad \varphi(y, z) \sim \rho(y^2 + yz - y^2), (y_0, z_0) = (0, 0) \pmod 1$$

where $\rho > 0$.

This is Theorem 3 of Blaney (1950).

3. Ternary Forms : Proof of Theorem 3

Let
$$\left.\begin{array}{l} m = \operatorname{Inf} -Q(u, v, w) \\ u, v, w \in \mathbf{Z} \\ Q(u, v, w) < 0. \end{array}\right\} \quad \ldots(3.1)$$

Then $m \geqslant 0$.

If $m = 0$, then (1.4) follows from Lemma 1 with $\alpha = \mu(f(\mu) \mid D \mid)^{1/3} + \delta$ where $0 < \delta < \frac{1}{2}(1 - \mu)(f(\mu) \mid D \mid)^{1/3}$.

Lemma 6 — If $Q \sim -\rho Q_i$, $\rho > 0$, $1 \leqslant i \leqslant 8$, $m > 0$, then (1.4) is satisfied.

Proof : Suppose without loss of generality that

$$Q = -Q_i, \; 1 \leqslant i \leqslant 8.$$

Let α_i be the coefficient of yz in Q_i for each i, $1 \leqslant i \leqslant 8$. Let D_i be the determinant of Q_i and let

$$d_i = (f(\mu) \mid D_i \mid)^{1/3}.$$

Therefore $(1 - \mu)^3 d_i^3 = (1 - \mu)^3 f(\mu) \mid D_i \mid \geqslant \dfrac{32}{3} \mid D_i \mid$.

Since $|D_1| = 1/4$, $|D_2| = 12$, $|D_3| = 192$, $|D_4| = 1/2$, $|D_5| = 9/2$, $|D_6| = 16$, $|D_7| = 81$, $|D_8| = 256$, we notice that

$$(1 - \mu)\, d_i > \alpha_i \text{ for each } i.$$

Now choose $x \equiv x_0 \pmod 1$ arbitrarily and $z \equiv z_0 \pmod 1$ such that $0 < z \leqslant 1$. Let $A_i = -Q_i(x, y, z) + \alpha_i yz$. Then A_i is a real number. Now choose $y \equiv y_0 \pmod 1$ such that

$$0 < A_i - \alpha_i yz - \mu d_i \leqslant \alpha_i z \leqslant \alpha_i < (1 - \mu)\, d_i$$

i.e.

$$\mu d_i < A_i - \alpha_i yz = -Q_i(x, y, z) < d_i.$$

Let now $m > 0$, $-Q \not\sim \rho Q_i$, $1 \leqslant i \leqslant 8$. Then given ϵ_0, $0 < \epsilon_0 < \frac{3}{4}$ there exist integers u, v, w such that

$$-Q(u, v, w) = m/(1 - \epsilon) \leqslant (\tfrac{4}{3} \mid D \mid)^{1/3} \qquad \text{...(3.2)}$$

where $0 \leqslant \epsilon < \epsilon_0$. Equality holds in (3.2) iff $-Q \sim \rho Q_9$ or ρQ_{10}. Further by the definition of m we must have g.c.d. $(u, v, w) = 1$. Applying a suitable unimodular transformation we can suppose that

$$-Q(1, 0, 0) = m/(1 - \epsilon)$$

and then write

$$Q(x, y, z) = -\{m/(1 - \epsilon)\}\{(x + hy + gz)^2 + \varphi(y, z)\}$$

where $\varphi(y, z)$ is an indefinite binary quadratic form with discriminant

$$\Delta^2 = 4 \mid D \mid \Big/ \left(\frac{m}{1 - \epsilon}\right)^3 \geqslant 4 \cdot \tfrac{3}{4} = 3$$

with equality iff $Q \sim -\rho Q_9$ or $-\rho Q_{10}$. Because of homogeneity it suffices to prove:

Theorem A — Let $Q(x, y, z) = -(x + hy + gz)^2 + \varphi(y, z)$ where $\varphi(y, z)$ is an indefinite binary quadratic form with discriminant

$$\Delta^2 = 4 \mid D \mid \geqslant 3 \qquad \text{...(3.3)}$$

with strict inequality unless $Q \sim -Q_9$ or $-Q_{10}$.

Let $d = (f(\mu) \mid D \mid)^{1/3}$, where $f(\mu)$ satisfies (1.5). Then given any real numbers x_0, y_0, z_0 there exist $(x, y, z) \equiv (x_0, y_0, z_0)$ (mod 1) such that

$$\mu d < Q(x, y, z) < d. \qquad \text{...(3.4)}$$

3.1. *Proof of Theorem A*

Remark 7 : One can easily verify that there is a real number α, $1/12 < \alpha < 1/9$ such that $f_1(\mu) \geqslant f_2(\mu)$ if and only if $\mu \leqslant \alpha$.

Lemma 8 — Let $Q(x, y, z)$ satisfy the conditions of Theorem A. Suppose that we can find $(y, z) \equiv (y_0, z_0)$ (mod 1) such that

$$\tfrac{1}{4} + \mu d < \varphi(y, z) \begin{cases} \leqslant \left(\dfrac{(1-\mu)\, d + 1}{2}\right)^2 + \mu d \text{ if } (1 - \mu)\, d \text{ is an integer} \\ < \left(\dfrac{[(1-\mu)\, d]}{2}\right)^2 + d \text{ if } (1 - \mu)\, d \text{ is not an integer.} \end{cases} \qquad \text{...(3.5)}$$

Then there exists $x \equiv x_0 \pmod 1$ satisfying

$$\mu d < Q(x, y, z) \leqslant d. \qquad \ldots(3.6)$$

Further strict inequality in (3.5) implies strict inequality in (3.6).

PROOF : Since $(1 - \mu)^3 d^3 = (1 - \mu)^3 f(\mu) \mid D \mid$

$$\geqslant (1 - \mu)^3 \frac{32}{3(1 - \mu)^3} \cdot \frac{3}{4} = 8,$$

the result follows from Lemma 4 with $\alpha = hy + gz$ and $\beta^2 = \varphi(y, z) - \mu d$ with d replaced by $(1 - \mu)d$ ($\geqslant 2$).

Lemma 9 — If $(1 - \mu)d = 2$, then (3.6) is true with strict inequality.

PROOF : One sees that $(1 - \mu)d = 2$ if and only if

$$f(\mu) = f_1(\mu), \mid D \mid = 3/4,$$

and $\qquad Q \sim -Q_9 = -2(x + \tfrac{1}{2}y)^2 - \tfrac{3}{2}y^2 + \tfrac{1}{4}z^2$

or $\qquad Q \sim -Q_{10} = -3x^2 - yz.$

By Remark 7, $\mu < 1/9$.

Case (i) : $Q = -2(x + \frac{1}{2}y)^2 + \frac{1}{4}z^2 - \frac{3}{2}y^2$. Choose $y \equiv y_0 \pmod 1$ such that $\mid y \mid \leqslant \frac{1}{2}$.

If $0 \leqslant \mid y \mid < \sqrt{\dfrac{1 - 5\mu}{3(1 - \mu)}}$, then choose $x \equiv x_0 \pmod 1$ such that $\mid x + \frac{1}{2}y \mid \leqslant \frac{1}{2}$ and $z \equiv z_0 \pmod 1$ such that $2 \leqslant \mid z \mid \leqslant \frac{5}{2}$. So that

$$\mu d = \frac{2\mu}{1 - \mu} = -\tfrac{2}{4} + \tfrac{4}{4} - \tfrac{3}{2} \cdot \frac{1 - 5\mu}{3(1 - \mu)} < -2(x + \tfrac{1}{2}y)^2 + \frac{z^2}{4} - \tfrac{3}{2}y^2$$

$$\leqslant \frac{z^2}{4} \leqslant \frac{25}{16} < d = \frac{2}{1 - \mu}.$$

If $\sqrt{\dfrac{1 - 5\mu}{3(1 - \mu)}} \leqslant \mid y \mid \leqslant \frac{1}{2}$, then choose $x \equiv x_0 \pmod 1$ such that $\mid x + \frac{1}{2}y \mid \leqslant \frac{1}{2}$ and $z \equiv z_0 \pmod 1$ such that $\frac{5}{2} \leqslant \mid z \mid \leqslant 3$. So that

$$\mu d = \frac{2\mu}{1 - \mu} < \frac{11}{16} = -\frac{2}{4} + \frac{25}{16} - \tfrac{3}{2}.\tfrac{1}{4} \leqslant -2(x + \tfrac{1}{2}y)^2 + \frac{z^2}{4} - \tfrac{3}{2}y^2$$

$$\leqslant \frac{z^2}{4} - \tfrac{3}{2}y^2$$

$$\leqslant \tfrac{9}{4} - \tfrac{3}{2}.\frac{1 - 5\mu}{3(1 - \mu)} = \frac{7 + \mu}{4(1 - \mu)} < \frac{2}{1 - \mu}.$$

Case (*ii*) — When $Q = -3x^2 - yz$, the proof is similar to that of Lemma 6.

3.2. Let now $(1 - \mu)\, d > 2$.

Let $n < (1 - \mu)d \leqslant n + 1, n = 2, 3, \ldots$ . ...(3.7)

Now we shall prove that there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$\tfrac{1}{4} + \mu d < \varphi(y, z) \leqslant \frac{n^2}{4} + d. \qquad \ldots(3.8)$$

Let $0 < \nu < 1$ be a real number satisfying

$$\frac{\nu^2 \Delta}{\{(1-\nu)^3 (1 + 3\nu)\}^{1/2}} = \tfrac{1}{4} + \mu d \qquad \ldots(3.9)$$

i.e. $$g(\nu) = (1-\nu)^3 (1 + 3\nu) - \frac{\Delta^2 \nu^4}{(\frac{1}{4} + \mu d)^2} = 0.$$

Such a choice of ν is always possible as $g(0) > 0$ and $g(1) < 0$. Then (3.8) will follow from Lemma 5 if we can show that

$$\frac{\Delta}{\{(1-\nu)^3 (1 + 3\nu)\}^{1/2}} \leqslant \frac{n^2}{4} + d$$

i.e. $$\frac{\frac{1}{4} + \mu d}{\nu^2} \leqslant \frac{n^2}{4} + d.$$

So it suffices to prove that

$$\nu^2 \geqslant \frac{1 + 4\mu d}{n^2 + 4d} = a^2 \text{ (say)}. \qquad \ldots(3.10)$$

Clearly $a^2 > \mu$ iff $a^2 < 1/n^2$, iff $\mu < 1/n^2$. ...(3.11)

Also if $\mu \neq \dfrac{1}{n^2}$, then $d = \dfrac{1 - a^2 n^2}{4(a^2 - \mu)}$. ...(3.12)

If $\mu = 1/n^2$, we have $a^2 = 1/n^2$. We discuss these cases separately.

Lemma 10 — If $n < (1-\mu)\, d \leqslant n + 1$, $\mu \neq 1/n^2$, then (3.10) is true.

PROOF : Since $h(\nu) = \dfrac{(1-\nu)^3 (1 + 3\nu)}{\nu^4} = \left(\dfrac{1}{\nu} - 1\right)^3 \left(\dfrac{1}{\nu} + 3\right)$ is a decreasing function of ν in $0 < \nu < 1$, to prove $\nu^2 \geqslant a^2$ it suffices to prove that $h(a) \geqslant h(\nu)$

i.e. $$\frac{(1-a)^3 (1 + 3a)}{a^4} \geqslant h(\nu) = \frac{\Delta^2}{(\frac{1}{4} + \mu d)^2} \text{ [from (3.9)]}$$

$$= \frac{(1 - n^2 a^2)^3}{f(\mu)\,(a^2 - \mu)\, a^4 (1 - n^2 \mu)^2}$$

$\left[\text{substituting for } \Delta^2 = \dfrac{4d^3}{f(\mu)} \text{ and } d \text{ from (3.12)}\right]$

i.e. $$\psi(a) = \frac{(1-a)^3(1+3a)(a^2-\mu)}{(1-n^2a^2)^3} \geqslant \frac{1}{f(\mu)(1-n^2\mu)^2}. \quad \ldots(3.13)$$

Since $$a^2 = \frac{1+4\mu d}{n^2+4d} = \frac{1-n^2\mu}{n^2+4d} + \mu,$$

as a function of d, a^2 decreases if $\mu < 1/n^2$ and increases if $\mu > 1/n^2$. Since

$$\frac{n}{1-\mu} < d \leqslant \frac{n+1}{1-\mu}$$

it follows that

$$\left.\begin{aligned} a^2 &\geqslant \frac{1+4\mu\dfrac{n+1}{1-\mu}}{n^2+4\,.\dfrac{n+1}{1-\mu}} = \frac{(4n+3)\mu+1}{(n+2)^2-n^2\mu} \quad \text{if } \mu < \frac{1}{n^2} \\ \text{and} \quad a^2 &\leqslant \frac{1+4\mu\dfrac{n+1}{1-\mu}}{n^2+4\,.\dfrac{n+1}{1-\mu}} = \frac{(4n+3)\mu+1}{(n+2)^2-n^2\mu} \quad \text{if } \mu > \frac{1}{n^2}. \end{aligned}\right\} \quad \ldots(3.14)$$

Differentiating $\psi(a)$ with respect to a, and using (3.11) one can verify that $\psi(a)$ increases for $0 < a < 1/n$ and decreases for $1/n < a < 1$. Therefore

$$\psi(a) \geqslant \psi\left(\sqrt{\frac{(4n+3)\mu+1}{(n+2)^2-n^2\mu}}\right)$$
$$= \frac{(\sqrt{(n+2)^2-n^2\mu}-\sqrt{(4n+3)\mu+1})^3(\sqrt{(n+2)^2-n^2\mu}+3\sqrt{(4n+3)\mu+1})(1-\mu)}{(4n+4)^3(1-n^2\mu)^2}.$$

Therefore (3.13) will be satisfied if

$$f(\mu) \geqslant \frac{4^3(n+1)^3}{(1-\mu)}\Big/(\sqrt{(n+2)^2-n^2\mu}-\sqrt{(4n+3)\mu+1})^3$$
$$\times(\sqrt{(n+2)^2-n^2\mu}+3\sqrt{(4n+3)\mu+1}).$$

Rationalizing and simplifying the right-hand side of (3.15) we see that (3.13) is true if

$$f(\mu) \geqslant \frac{4^3}{(n+3)^3}\cdot\frac{1}{(1-\mu)^4}\,\frac{(\sqrt{(n+2)^2-n^2\mu}+\sqrt{(4n+3)\mu+1})^3}{(\sqrt{(n+2)^2-n^2\mu}+3\sqrt{(4n+3)\mu+1})}.$$
$$= f(n,\mu) \text{ (say)} \quad \ldots(3.16)$$

One can prove that for fixed μ, $0 \leqslant \mu < 1$, $f(n, \mu)$ is a strictly decreasing function of n for $n \geqslant 2$. Also $f(2, \mu) = f_2(\mu)$. Therefore (3.13) will be satisfied if $f(\mu) \geqslant f_2(\mu)$ and equality in (3.8) can occur only if $n = 2$, $f(\mu) = f_2(\mu)$, $d = \dfrac{n+1}{1-\mu} = \dfrac{3}{1-\mu}$ and $\nu = a$. Hence the result follows in this case.

Lemma 11 — If $n < (1-\mu)d \leqslant n+1$, and $\mu = 1/n^2$, then (3.10) is true.

PROOF: $\mu = \frac{1}{n^2}$ implies $a^2 = \frac{1+4\mu d}{n^2+4\,.\,d} = \frac{1}{n^2}$.

As in Lemma 10, one sees that to prove $\nu^2 > a^2 = 1/n^2$ it suffices to prove that

$$h\left(\frac{1}{n}\right) = \frac{\left(1-\frac{1}{n}\right)^3\left(1+\frac{3}{n}\right)}{\left(\frac{1}{n}\right)^4} \geqslant h(\nu) = \frac{64d^3}{f\left(\frac{1}{n^2}\right)\left(1+\frac{4}{n^2}d\right)^2}$$

i.e.
$$g(d) = \frac{(n^2+4d)^2}{d^3} \geqslant \frac{64n^4}{f\left(\frac{1}{n^2}\right)(n-1)^3(n+3)}.$$

$g(d)$ is a decreasing function of d and $\frac{n}{1-\mu} < d \leqslant \frac{n+1}{1-\mu}$, therefore

$$g(d) \geqslant g\left(\frac{n+1}{1-\mu}\right) = g\left(\frac{n^2}{n-1}\right) = \frac{(n-1)(n+3)^2}{n^2}.$$

This will be $\geqslant \frac{64n^4}{f\left(\frac{1}{n^2}\right)(n-1)^3(n+3)}$

if
$$f\left(\frac{1}{n^2}\right) \geqslant \frac{64n^6}{(n-1)^4(n+3)^3} = f\left(n, \frac{1}{n^2}\right)$$

where $f(n, \mu)$ is as defined in (3.16). Since $f(\mu) \geqslant f(n, \mu)$ for all μ in $0 \leqslant \mu < 1$, we have in particular $f\left(\frac{1}{n^2}\right) \geqslant f\left(n, \frac{1}{n^2}\right)$.

Thus the Lemma is proved.

Remark: Equality can occur in (3.10) only if $f(\mu) = f_2(\mu)$, $d = \frac{n+1}{1-\mu} = \frac{3}{1-\mu}$ and $\nu = a$.

Lemma 12 — (3.6) holds with strict inequality, so that theorem A follows.

PROOF: If we have strict inequality in (3.8) we have strict inequality in (3.6) by Lemma 8. Therefore suppose that equality holds in (3.8). As remarked above, it happens only if

$$f(\mu) = f_2(\mu),\ d = \frac{3}{1-\mu},\ n = 2,$$

$$a^2 = \frac{(4n+3)\mu+1}{(n+2)^2-n^2\mu} = \frac{11\mu+1}{16-4\mu} \text{ and } \nu^2 = a^2.$$

Further by Lemma 5, equality can occur only at $\nu = 0, \frac{1}{3}, \frac{1}{2}$. $\nu = 0$ is not possible as $\nu^2 = \frac{11\mu + 1}{16 - 4\mu} \neq 0$. $\nu = \frac{1}{3}$ is also not possible as $\nu^2 = \frac{11\mu + 1}{16 - 4\mu} = \frac{1}{9}$ implies $\mu = \frac{7}{103}$ and $f(\mu) > f_2(\mu)$ for $0 \leqslant \mu < \frac{1}{12}$. So let $\nu = \frac{1}{2}$, then $\frac{11\mu + 1}{16 - 4\mu} = \frac{1}{4}$ implies $\mu = \frac{1}{4}$ and therefore $d = \frac{3}{1 - \mu} = 4$.

From Lemma 5, we can suppose

$$\varphi(y, z) = \rho(y^2 + yz - z^2), (y_0, z_0) \equiv (0, 0) \pmod 1$$

$$5\rho^2 = \Delta^2 = \frac{4d^3}{f(\mu)} = \frac{4 \cdot 4^3}{f_2(\frac{1}{4})} = \frac{5^3}{4^2}.$$

Therefore

$$\rho = \tfrac{5}{4}$$

and $\varphi(y, z) = \frac{5}{4}(y^2 + yz - z^2), (y_0, z_0) \equiv (0, 0) \pmod 1$.

Let $F(x, y, z) = -(x + x_0 + hy + gz)^2 + \frac{5}{4}(y^2 + yz - z^2)$.

Take $y = 1, z = 1$, and choose integer x such that

$$| x + x_0 + h + g | \leqslant \tfrac{1}{2}.$$

Then

$$\mu d = 1 = -\tfrac{1}{4} + \tfrac{5}{4} < F(x, 1, 1)$$
$$= -(x + x_0 + h + g)^2 + \tfrac{5}{4} \leqslant \tfrac{5}{4} < 4 = d.$$

Therefore we have strict inequality in (3.6) unless

$$x_0 + h + g \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(3.17)}$$

Similarly considering $F(x, -1, 0)$ and $F(x, 1, 0)$ we have strict inequality unless

$$x_0 - h \equiv \tfrac{1}{2} \pmod 1 \qquad \text{...(3.18)}$$

$$x_0 + h \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(3.19)}$$

From (3.17), (3.18), (3.19) we must have

$$g \equiv 0 \pmod 1 \text{ and } 2h \equiv 0 \pmod 1.$$

Replacing x by $x + ry + sz$ where r and s are suitable integers we can suppose that

$$| h | \leqslant \tfrac{1}{2}, | g | \leqslant \tfrac{1}{2}.$$

Therefore we must have

$$g = 0 \text{ and } h = 0 \text{ or } \pm \tfrac{1}{2}.$$

If $h = 0$, then $x_0 \equiv \frac{1}{2} \pmod 1$.

When $x_0 = \frac{1}{2}$, we observe that

$$1 < F(1, 2, 0) < 4.$$

If $h = \pm \frac{1}{2}$, then $x_0 \equiv 0 \pmod 1$.

When $x_0 = 0$, we observe that

$$1 < F(1, 2, 1) < 4 \quad \text{if} \quad h = \tfrac{1}{2}$$
$$1 < F(3, 2, 1) < 4 \quad \text{if} \quad h = -\tfrac{1}{2}.$$

So we always have strict inequality in (3.6). This proves Theorem A and Theorem 1 now follows from Lemmas 6-12.

4. Quaternary Forms : Proof of Theorem 1

Let
$$\left.\begin{aligned} m' &= \operatorname{Inf} -Q(x, y, z, t) \\ &\quad x, y, z, t \text{ integers} \\ &\quad Q(x, y, z, t) < 0. \end{aligned}\right\} \qquad \ldots(4.1)$$

Then $m' \geqslant 0$.

Case I — If $m' = 0$ then the result follows from Lemma 1 with $\alpha = \delta$ and $0 < \delta < \frac{1}{2} (K \mid D \mid)^{1/4}$.

Case II — If $m' > 0$, then given $0 < \epsilon_0 < \frac{3}{4}$, there exist integers x_1, y_1, z_1, t_1, such that

$$-Q(x_1, y_1, z_1, t_1) = \frac{m'}{1 - \epsilon} \leqslant \left(\frac{16}{3} \mid D \mid \right)^{1/4} \qquad \ldots(4.2)$$

where $0 \leqslant \epsilon < \epsilon_0$. By the definition of m' we must have g.c.d. $(x_1, y_1, z_1, t_1) = 1$. Replacing $Q(x, y, z, t)$ by an equivalent form we can suppose that

$$-Q(1, 0, 0, 0) = \frac{m'}{1 - \epsilon}$$

and write

$$Q(x, y, z, t) = -\frac{m'}{1 - \epsilon} \{(x + hy + gz + vt)^2 - \varphi(y, z, t)\}$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form of the type (1, 2) and determinant $\mid D \mid \Big/ \left(\frac{m'}{1 - \epsilon} \right)^4 \geqslant 3/16$.

Because of homogeneity it suffices to prove.

Theorem B — Let $Q(x, y, z, t) = -(x + hy + gz + vt)^2 + \varphi(y, z, t)$ where $\varphi(y, z, t)$ is an indefinite ternary quadratic form of the type (1, 2) and determinant $-D = \mid D \mid > 3/16$.

Let $d = (K\,|\,D\,|\,)^{1/4}$ and $K = \dfrac{128}{25}(2\sqrt{7} - 1)$. ...(4.3)

Then given any real numbers x_0, y_0, z_0, t_0, we can find $(x, y, z, t) \equiv (x_0, y_0, z_0, t_0)$ (mod 1) such that

$$0 < Q(x, y, z, t) < d. \qquad ...(4.4)$$

Lemma 13 — Let $Q(x, y, z, t)$ satisfy the conditions of Theorem B. Suppose we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that

$$\tfrac{1}{4} < \varphi(y, z, t) < \begin{cases} \left(\dfrac{d+1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ \left(\dfrac{[d]}{2}\right)^2 + d & \text{if } d \text{ is not an integer.} \end{cases} \qquad ...(4.5)$$

Then there exists $x \equiv x_0$ (mod 1) satisfying (4.4).

PROOF: Since $d = (K\,|\,D\,|\,)^{1/4} > (24(2\sqrt{7} - 1)/25)^{1/4} > 1$ the proof follows from Lemma 4 and taking $\alpha = hy + gz + vt$ and $\beta^2 = \varphi(y, z, t)$.

Lemma 14 — If $d \geqslant 3$, then we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) satisfying (4.5).

PROOF: Since $\left(\dfrac{d+1}{2}\right)^2 < \left(\dfrac{[d]}{2}\right)^2 + d$, it suffices to prove that there exist $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that

$$\tfrac{1}{4} < \varphi(y, z, t) < \left(\frac{d+1}{2}\right)^2. \qquad ...(4.6)$$

By Theorem 3, applied to $\varphi(y, z, t)$ with $\mu = \dfrac{1}{(d+1)^2}$ (so that $0 \leqslant \mu \leqslant 1/16 < 1/12$) there exist $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that

$$\mu(f(\mu)\,|\,D\,|\,)^{1/3} < \varphi(y, z, t) < (f(\mu)\,|\,D\,|\,)^{1/3}$$

if $\qquad f(\mu) \geqslant f_1(\mu).$

(4.6) will follow if

$$\tfrac{1}{4} < \mu(f(\mu)\,|\,D\,|\,)^{1/3} \text{ and } \left(\frac{d+1}{2}\right)^2 \leqslant (f(\mu)\,|\,D\,|\,)^{1/3}$$

i.e. if $$\frac{1}{64\mu^3}\,|\,D\,|^{-1} \leqslant f(\mu) \leqslant \left(\frac{d+1}{2}\right)^6 |\,D\,|^{-1}.$$

This is satisfied if we take $f(\mu) = \left(\dfrac{d+1}{2}\right)^6 \dfrac{1}{|\,D\,|}$.

Therefore (4.6) will follow if we have

$$f\left(\frac{1}{(d+1)^2}\right) \geqslant f_1\left(\frac{1}{(d+1)^2}\right)$$

i.e. if $$\left(\frac{d+1}{2}\right)^6 \frac{1}{|D|} = \frac{(d+1)^6}{d^4} \cdot \frac{K}{64} \geqslant \frac{32}{3\left(1-\frac{1}{(d+1)^2}\right)^3}$$

i.e. if $$\frac{K}{64} \cdot \frac{3}{32} \geqslant \frac{d}{(d+2)^3}.$$

Right-hand side is a decreasing function of d and $d \geqslant 3$, therefore

$$\frac{d}{(d+2)^3} < \frac{3}{5^3} < \frac{K}{64} \cdot \frac{3}{32}$$

if $K > \frac{64 \, . \, 32}{125}$, which is true.

Lemma 15 — If $2 < d \leqslant 3$, then again we can find $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ satisfying (4.5).

PROOF: By Lemma 13, it is enough to prove that there exist $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ such that

$$\tfrac{1}{4} < \varphi(y, z, t) < 1 + d. \qquad ...(4.7)$$

We apply Theorem 3 to $\varphi(y, z, t)$ with $f(\mu) = \frac{1}{64\mu^3} \cdot \frac{1}{|D|}$ and $\mu = \frac{1}{4(1+d)}\left(< \frac{1}{12}\right)$. There exist $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ satisfying

$$\tfrac{1}{4} = \mu(f(\mu)\,|D|)^{1/3} < \varphi(y, z, t) < (f(\mu)\,|D|)^{1/3} = 1 + d$$

if $$f(\mu) \geqslant f_1(\mu) = \frac{32}{3(1-\mu)^3}.$$

Substituting for μ we see that this is so if

$$\frac{(3+4d)^3}{d^4} \geqslant \frac{32 \, . \, 4^3}{3.K}.$$

This can be easily verified.

Lemma 16 — If $1 < d \leqslant 2$, then again (4.5) is true.

PROOF: By Lemma 13, it is enough to prove that there exist $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ such that

$$\tfrac{1}{4} < \varphi(y, z, t) < \tfrac{1}{4} + d. \qquad ...(4.8)$$

Apply Theorem 3 to $\varphi(y, z, t)$ with $f(\mu) = \frac{1}{64\mu^3} \cdot \frac{1}{|D|}$ and $\mu = \frac{1}{1+4d}\left(\geqslant \frac{1}{9}\right)$
Then we can find $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ such that

$$\tfrac{1}{4} = \mu(f(\mu)\,|\,D\,|\,)^{1/3} < \varphi(y, z, t) < (f(\mu)\,|\,D\,|\,)^{1/3} = \tfrac{1}{4} + d$$

provided

$$f(\mu) \geqslant f_2(\mu) = \frac{4^3}{5^3} \cdot \frac{1}{(1-\mu)^4} \frac{\{(16-4\mu)^{1/2} + (11\mu+1)^{1/2}\}^3}{(16-4\mu)^{1/2} + 3(11\mu+1)^{1/2}}. \qquad \ldots(4.9)$$

Substituting for μ, we see that (4.9) is true if

$$\frac{\{(12+64d)^{1/2} + (12+4d)^{1/2}\}^3}{(12+64d)^{1/2} + 3(12+4d)^{1/2}} \leqslant \frac{125}{16}K.$$

Now L.H.S. is an increasing function of d and $d \leqslant 2$, therefore

$$\text{L.H.S.} \leqslant \frac{(\sqrt{140} + \sqrt{20})^3}{\sqrt{140} + 3\sqrt{20}} = 20 \,.\, \frac{10\sqrt{7} + 22}{\sqrt{7} + 3} = 40(2\sqrt{7} - 1) = \frac{125}{16}K.$$

This completes the proof of the lemma.

Thus Theorem B follows from Lemmas 13–16 and Theorem 1 is proved.

5. Proof of Theorem 2

$$\begin{aligned} Q(x, y, z, t) &= -x^2 - xz - y^2 - yz - z^2 + 2zt \\ &= -(x + \tfrac{1}{2}z)^2 - (y + \tfrac{1}{2}z)^2 - \tfrac{1}{2}(z - 2t)^2 + 2t^2 \end{aligned}$$

is an indefinite quaternary quadratic form of the type (1, 3) and determinant $D = -1$.

It can be easily verified that the inequality

$$\begin{aligned} 0 < Q(x, y, z, t + \tfrac{1}{2}) &= -x^2 - xz - y^2 - yz - z^2 + z(2t+1) \\ &< 2 = (16\,|\,D\,|\,)^{1/4} \end{aligned}$$

has no solution in integers x, y, z, and t.

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