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INHOMOGENEOUS MINIMUM OF INDEFINITE QUADRATIC FORMS IN FIVE VARIABLES OF TYPE (4, 1) OR (1, 4): A CONJECTURE OF WATSON

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In the present paper it is proved that if $Q(x, y, z, t, u)$ is a real indefinite quadratic form of type (4, 1) or (1, 4) and determinant D, then for any real numbers x_0, y_0, z_0, t_0, u_0 there exist integers x, y, z, t, u such that

$$| Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) | \leqslant (\tfrac{1}{2} | D |)^{1/5}.$$

All the critical forms are also obtained.

1. INTRODUCTION

Let $Q(x_1, x_2, ..., x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n - r)$, $0 < r < n$ and determinant $D \neq 0$. Blaney (1948) proved that there exist constants C such that for any such quadratic form Q and any real numbers $c_1, c_2, ..., c_n$ we can find integers $x_1, x_2, ..., x_n$ satisfying

$$| Q(x_1 + c_1, x_2 + c_2, ..., x_n + c_n) | \leqslant (C | D |)^{1/n}. \qquad ...(1.1)$$

Let $C_{r,n-r}$ denote the greatest lower bound of all such constants C. $C_{1,1} = \frac{1}{4}$ constitutes the classical result of Minkowski. $C_{2,1} = C_{1,2} = 27/100$ was proved by Davenport (1948). Birch (1953) showed that $C_{r,r} = \frac{1}{4}$ for all $r \geqslant 1$. $C_{1,3} = C_{3,1} = \frac{1}{3}$ was proved by Dumir (1967). Authors (Hans-Gill and Madhu Raka 1980) showed that $C_{3,2} = C_{2,3} = \frac{1}{4}$. In this paper we shall prove that $C_{1,4} = C_{4,1} = \frac{1}{2}$; proving a conjecture of Watson (1962). This completes the case $n = 5$. All the critical forms for which the sign of equality is necessary, are also obtained. More precisely we prove:

Theorem — Let $Q(x, y, z, t, u)$ be a real indefinite quadratic form in five variables of the type (4, 1) or (1, 4) and determinant $D \neq 0$. Then given any real numbers x_0, y_0, z_0, t_0, u_0 we can find integers x, y, z, t, u such that

$$| Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) | \leqslant (\tfrac{1}{2} | D |)^{1/5}. \qquad ...(1.2)$$

Equality is needed if and only if

$$Q(x, y, z, t, u) \sim \rho Q_1 = (x^2 + y^2 + 2z^2 + 2tu), \ \rho \neq 0 \qquad ...(1.3)$$

and for Q_1 equality is necessary if and only if

$$(x_0, y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \pmod 1.$$

2. Some Lemmas

In the course of the proof we shall use the following lemmas.

Lemma 1 — Let $Q(x_1, ..., x_n)$ be an indefinite quadratic form in $n(\geqslant 3)$ variables with real coefficients. Suppose that Q is non-singular and takes arbitrarily small non-zero values for integral $x_1, ..., x_n$. Let $c_1, ..., c_n, \alpha, \delta$ be real numbers, $\delta > 0$. Then we can find integers $x_1, x_2, ..., x_n$ such that

$$| Q(x_1 + c_1, ..., x_n + c_n) - \alpha | < \delta. \quad ...(2.1)$$

This is Theorem 1 of Watson (1960).

Lemma 2 — If $Q(x, y, z, t, u)$ is an indefinite quadratic form of the type (4, 1) and determinant $D(< 0)$, then there exist integers x_1, y_1, z_1, t_1, u_1 such that

$$0 < Q(x_1, y_1, z_1, t_1, u_1) \leqslant (8 \mid D \mid)^{1/5}. \quad ...(2.2)$$

This follows from results of Watson (1968), Oppenheim (1953a) and Jackson (1969).

Lemma 3 — Let α, β, d be real numbers with $d > \frac{1}{4}$. Then given any real number x_0 there exists $x \equiv x_0 \pmod 1$ such that

$$| (x + \alpha)^2 - \beta^2 | \leqslant d \quad ...(2.3)$$

provided

$$\beta^2 \leqslant \begin{cases} d^2 + 1/4 & \text{if } 2d \text{ is an integer} \\ 1/4\,([2d])^2 + d & \text{if } 2d \text{ is not an integer.} \end{cases} \quad ...(2.4)$$

Further strict inequality in (2.4) implies strict inequality in (2.3).

This is Lemma 5 of Davenport (1948).

Lemma 4 — Let $\varphi(y, z, t, u)$ be an indefinite quaternary quadratic form of the type (1, 3) and determinant $D(< 0)$. Then there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ (mod 1) such that

$$0 < \varphi(y, z, t, u) < (22 \mid D \mid)^{1/4}. \quad ...(2.5)$$

This follows from a Theorem of Hans-Gill and Madhu Raka (1980).

Lemma 5 — If $\varphi(y, z, t, u)$ is an indefinite quaternary quadratic form of the type (3, 1) and determinant $D(< 0)$ then there exist integers y_2, z_2, t_2, u_2 such that

$$0 < \varphi(y_2, z_2, t_2, u_2) \leqslant \left(\frac{2048}{729} \mid D \mid\right)^{1/4} \quad ...(2.6)$$

except when

$$\varphi(y, z, t, u) \sim \rho\varphi_2 = \rho(y^2 + yz + z^2 + tu)$$

and

$$\varphi(y, z, t, u) \sim \rho\varphi_3 = \rho(y^2 + z^2 + tu)$$

where $\rho > 0$.

This is Theorem 2 of Oppenheim (1953b).

Lemma 6 — Let α, β, d be real numbers with $d \geqslant 1$. Then given any real number y_0 there exists $y \equiv y_0 \pmod 1$ such that

$$0 < (y + \alpha)^2 - \beta^2 \leqslant d \qquad \text{...(2.7)}$$

provided

$$\beta^2 \begin{cases} \leqslant \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ < \left(\dfrac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer.} \end{cases} \qquad \text{...(2.8)}$$

Further strict inequality in (2.8) implies strict inequality in (2.7).

This is Lemma 6 of Dumir (1968).

Lemma 7 — Let $3 \leqslant \nu < \infty$ be a real number. Let $\psi(z, t, u)$ be an indefinite ternary quadratic form of the type (2, 1) and determinant $D(< 0)$. Then given any real numbers z_0, t_0, u_0 there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\nu \left(\frac{|D|}{(1+\nu)^2}\right)^{1/3} < \psi(z, t, u) \leqslant \left(\frac{|D|}{(1+\nu)^2}\right)^{1/3}. \qquad \text{...(2.9)}$$

Equality occurs only at $\nu = 3$ and 7.

At $\nu = 3$, equality is needed if and only if

$$\psi(z, t, u) \sim \rho\psi_4 = \rho(z^2 + tu),\ (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, 0, 0) \pmod 1$$

or

$$\psi(z, t, u) \sim \rho\psi_5 = \rho(2z^2 + t^2 - u^2),\ (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1$$

where $\rho > 0$.

This follows from a theorem of Dumir (1969).

3. Proof of the Theorem

Replacing Q by $-Q$, if necessary, we can suppose that Q is of the type (4, 1).

Let

$$\left.\begin{aligned} m &= \operatorname{Inf} Q(x, y, z, t, u) \\ &\quad x, y, z, t, u \text{ integers} \\ &\quad Q(x, y, z, t, u) > 0. \end{aligned}\right\} \qquad \text{...(3.1)}$$

Then $m \geqslant 0$.

If $m = 0$ then the result follows from Lemma 1 on taking $\alpha = 0$ and $\delta = (\frac{1}{2} \mid D \mid)^{1/5}$.

So let now $m > 0$.

Then given $0 < \epsilon_0 < 1/16$, we can find integers x_1, y_1, z_1, t_1, u_1 to satisfy

$$Q(x_1, y_1, z_1, t_1, u_1) = m/(1 - \epsilon)$$

where $0 \leqslant \epsilon < \epsilon_0$. Further by Lemma 2 we can assume that

$$Q(x_1, y_1, z_1, t_1, u_1) = m/(1 - \epsilon) \leqslant (8 \mid D \mid)^{1/5}. \qquad \ldots(3.2)$$

Since $0 \leqslant \epsilon < \epsilon_0 < 1/16$, by the definition of m we must have g.c.d. $(x_1, y_1, z_1, t_1, u_1) = 1$. By a suitable unimodular transformation we can suppose that

$$Q(1, 0, 0, 0, 0) = m/(1 - \epsilon)$$

and write

$$Q(x, y, z, t, u) = \frac{m}{1 - \epsilon}\{(x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)\}$$

where $\mid h \mid \leqslant \frac{1}{2}$, $\mid g \mid \leqslant \frac{1}{2}$, $\mid h' \mid \leqslant \frac{1}{2}$, $\mid g' \mid \leqslant \frac{1}{2}$ and $\varphi(y, z, t, u)$ is an indefinite quadratic form of the type (3, 1) and determinant

$$\mid D \mid /[m/(1 - \epsilon)]^5 \geqslant 1/8 \quad \text{(using (3.2))}.$$

Also by the definition of m we have for all integers x, y, z, t, u either

$$Q(x, y, z, t, u) \leqslant 0 \quad \text{or} \quad Q(x, y, z, t, u) \geqslant m$$

i.e. either $\quad (x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u) \leqslant 0$

or $\quad (x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u) \geqslant 1 - \epsilon.$

Because of homogeneity it suffices to prove:

Theorem A — Let $Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)$ where $\varphi(y, z, t, u)$ is an indefinite quaternary quadratic form of the type (3, 1) and determinant D such that

$$\mid D \mid \geqslant \tfrac{1}{8} \qquad \ldots(3.3)$$

and

$$\mid h \mid \leqslant \tfrac{1}{2}, \ \mid g \mid \leqslant \tfrac{1}{2}, \ \mid h' \mid \leqslant \tfrac{1}{2}, \ \mid g' \mid \leqslant \tfrac{1}{2}. \qquad \ldots(3.4)$$

Suppose further that for integers x, y, z, t, u we have either

$$Q(x, y, z, t, u) \leqslant 0 \quad \text{or} \quad Q(x, y, z, t, u) \geqslant 1 - \epsilon. \qquad \ldots(3.5)$$

where $0 \leqslant \epsilon < 1/16$. Let

$$d = (\tfrac{1}{2} \mid D \mid)^{1/5}. \qquad \ldots(3.6)$$

Then given any real numbers x_0, y_0, z_0, t_0, u_0 we can find $(x, y, z, t, u) \equiv (x_0, y_0, z_0, t_0, u_0) \pmod 1$ such that

$$| Q(x, y, z, t, u) | \leqslant d. \qquad \text{...(3.7)}$$

Equality in (3.7) is needed if and only if Q is equivalent to the form Q_1 given by (1.3).

Proof of Theorem A

We observe that

$$d = (\tfrac{1}{2} \mid D \mid)^{1/5} \geqslant (\tfrac{1}{2} \cdot \tfrac{1}{8})^{1/5} \ (> \tfrac{1}{2}). \qquad \text{...(3.8)}$$

Lemma 8 — Let $\nu_1 = d - \frac{1}{4}$ and $\nu_2 > 0$ be such that

$$\nu_2 \leqslant \begin{cases} d^2 + \frac{1}{4} \text{ if } 2d \text{ is an integer} \\ d + \frac{1}{4}([2d])^2 \text{ if } 2d \text{ is not an integer.} \end{cases}$$

Suppose that there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ satisfying

$$-\nu_2 < \varphi(y, z, t, u) \leqslant \nu_1. \qquad \text{...(3.9)}$$

Then for any x_0 there exists $x \equiv x_0 \pmod 1$ satisfying (3.7).

Further strict inequality in (3.9) implies strict inequality in (3.7).

PROOF : If $0 < \varphi(y, z, t, u) \leqslant \nu_1$, then choose $x \equiv x_0 \pmod 1$ such that $| x + hy + gz + h't + g'u | \leqslant \frac{1}{2}$, so that

$$0 < Q(x, y, z, t, u) \leqslant \nu_1 + \tfrac{1}{4} = d$$

If $-\nu_2 < \varphi(y, z, t, u) \leqslant 0$, then apply Lemma 3 with $\alpha = hy + gz + h't + g'u$ and $\beta^2 = -\varphi(y, z, t, u)$ to get the desired result.

Remark : $d^2 + \frac{1}{4} < d + \frac{1}{4}([2d])^2$, so that if (3.9) is satisfied with $\nu_2 \leqslant d^2 + \frac{1}{4}$, then (3.7) has a solution.

Lemma 9 — If $d > 3$, then (3.7) is true with strict inequality.

PROOF : By Lemma 4, applied to $-\varphi(y, z, t, u)$, we can find $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

$$-(22 \mid D \mid)^{1/4} < \varphi(y, z, t, u) < 0. \qquad \text{...(3.10)}$$

Then (y, z, t, u) will satisfy (3.9) with strict inequality if we have

$$(22 \mid D \mid)^{1/4} = (44d^5)^{1/4} < d^2 + \tfrac{1}{4}$$

i.e. if

$$f(d) = (d^2 + \tfrac{1}{4})^4 - 44d^5 > 0. \qquad \text{...(3.11)}$$

$f(d)$ is an increasing function of d for $d \geqslant 7/2$ and $f(7/2) > 0$. Hence (3.11) is satisfied for $d \geqslant 7/2$.

If $3 < d < 7/2$, then $[2d] = 6$. In this case (3.10) implies (3.9) if we have

$$(22 \mid D \mid)^{1/4} < d + 9.$$

One can see that this is true for $3 < d < 7/2$.

Remark : For $d \leqslant 3$, we shall apply the procedure of reduction, described in §3, to the quaternary forms of the type (3, 1). Here we shall use Lemma 5 on the homogeneous minima of (3, 1) forms. So we dispose of the forms φ_2 and φ_3 first.

Lemma 10 — If $\varphi(y, z, t, u) \sim \rho\varphi_2 = \rho(y^2 + yz + z^2 + tu)$, $\rho > 0$, $\frac{1}{2} < d \leqslant 3$, then (3.7) is true with strict inequality.

PROOF : Without loss of generality we can suppose that

$$\varphi(y, z, t, u) = \rho(y^2 + yz + z^2 + tu).$$

So that

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \rho(y^2 + yz + z^2 + tu).$$

Since from (3.4), $\mid g' \mid \leqslant 1/2$, and $Q(0, 0, 0, 0, 1) = g'^2 \leqslant 1/4 < 1 - \epsilon$, we must have, from (3.5), $g' = 0$. Similarly, considering $Q(0, 0, 0, 1, 0)$, $Q(0, 0, 1, -1, 1)$ and $Q(0, 1, 0, 1, -1)$ we get $h' = g = h = 0$. Therefore

$$Q(x, y, z, t, u) = x^2 + \rho(y^2 + yz + z^2 + tu).$$

Then

$$\mid D \mid = \rho^4 \cdot \frac{3}{16} \text{ i.e. } \rho^4 = \frac{16}{3} \mid D \mid = \frac{32}{3} d^5.$$

Case (i) : $d < \frac{3}{2}$

Choose $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ arbitrarily and $t \equiv t_0 \pmod 1$ such that $0 < t \leqslant 1$. Now choose $u \equiv u_0 \pmod 1$ satisfying

$$\mid x^2 + \rho(y^2 + yz + z^2 + tu) \mid \leqslant \rho \mid t \mid /2.$$

Since $\rho^4 = \frac{32}{3} d^5 < 16 d^4$ for $d < 3/2$, we get

$$\rho \mid t \mid /2 \leqslant \rho/2 < d.$$

Case (ii) : $\frac{3}{2} \leqslant d \leqslant 3$

In this case $\rho^4 = \frac{32}{3} d^5 \geqslant \frac{32}{3} \left(\frac{3}{2}\right)^5 = 3^4$ implies $\rho \geqslant 3$. Also one can verify that $\rho < 4d$.

If $(t_0, u_0) \not\equiv (0, 0) \pmod 1$ we can suppose without loss of generality that $t_0 \not\equiv 0 \pmod 1$. Then choose x, y, z arbitrarily, $t \equiv t_0 \pmod 1$ such that $0 < | t | \leqslant \frac{1}{2}$ and $u \equiv u_0 \pmod 1$ such that

$$| x^2 + \rho(y^2 + yz + z^2 + tu) | \leqslant \rho \, | t | /2 \leqslant \rho/4 < d.$$

If $(t_0, u_0) \equiv (0, 0) \pmod 1$, take $t = 1, u = -1$, and choose $z \equiv z_0 \pmod 1$ such that $\frac{1}{2} \leqslant | z | \leqslant 1$. Then choose $y \equiv y_0 \pmod 1$ such that $| y + \frac{1}{2} z | \leqslant \frac{1}{2}$, so we have

$$-\frac{13}{16}\rho = \frac{3}{4}\rho \cdot \frac{1}{4} - \rho \leqslant \varphi(y, z, 1, -1) = \rho\left(y + \frac{1}{2}z\right)^2 + \frac{3}{4}\rho z^2 - \rho$$

$$\leqslant \frac{1}{4}\rho + \frac{3}{4}\rho - \rho = 0.$$

Then (3.9) will be satisfied if we have

$$\frac{13}{16}\rho < d^2 + \frac{1}{4} = \left(\frac{3}{32}\rho^4\right)^{2/5} + \frac{1}{4}$$

i.e. if $$g(\rho) = \frac{(13\rho - 4)^5}{\rho^8} < 9.4^5.$$

$g(\rho)$ is a decreasing function of ρ for $\rho \geqslant 3$. Therefore

$$g(\rho) \leqslant g(3) = \frac{35^5}{3^8} < 9.4^5.$$

Hence case (ii) follows from Lemma 8.

Lemma 11 — If $\varphi(y, z, t, u) \sim \rho\varphi_3 = \rho(y^2 + z^2 + tu)$, $\rho > 0$, $\frac{1}{2} < d \leqslant 3$, then again (3.7) is true with strict inequality.

PROOF : Without loss of generality we can suppose that

$$\varphi(y, z, t, u) = \rho(y^2 + z^2 + tu).$$

So that

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \rho(y^2 + z^2 + tu).$$

As in the proof of Lemma 10, we see that (3.5) is contradicted unless we have $h = g = h' = g' = 0$. Therefore

$$Q(x, y, z, t, u) = x^2 + \rho(y^2 + z^2 + tu).$$

Then $| D | = \rho^4/4$ and so $\rho^4 = 4 \, | D | = 8d^5$.

Case (*i*) : $d < 2$

We choose $(x, y, z, t) \equiv (x_0, y_0, z_0, t_0) \pmod 1$ as in case (i) of Lemma 10. Then choose $u \equiv u_0 \pmod 1$ to satisfy

$$| x^2 + \rho(y^2 + z^2 + tu) | \leqslant \rho \mid t \mid /2.$$

since $\rho^4 = 8\, d^5 < 16\, d^4$ for $d < 2$, we have $\rho \mid t \mid /2 \leqslant \rho/2 < d$.

Case (ii): $2 \leqslant d \leqslant 3$

In this case $\rho^4 = 8d^5 \geqslant 8.2^5$ and so $\rho \geqslant 4$. Also $\rho < 4d$.

If $(t_0, u_0) \not\equiv (0, 0) \pmod 1$ we can suppose without loss of generality $t_0 \not\equiv 0 \pmod 1$. Then choose $t \equiv t_0 \pmod 1$ such that $0 < \mid t \mid \leqslant \frac{1}{2}$. Choose x, y, z, u as in case (i), so that

$$| x^2 + \rho(y^2 + z^2 + tu) | \leqslant \rho \mid t \mid /2 \leqslant \rho/4 < d.$$

If $(t_0, u_0) \equiv (0, 0) \pmod 1$, take $t = 1$, $u = -1$, choose $z \equiv z_0 \pmod 1$ such that $\mid z \mid \leqslant \frac{1}{2}$ and $y \equiv y_0 \pmod 1$ such that $\mid y \mid \leqslant \frac{1}{2}$, so that

$$-\rho \leqslant \varphi(y, z, 1, -1) \leqslant \rho(\tfrac{1}{4} + \tfrac{1}{4} - 1) < 0 < d - \tfrac{1}{4}.$$

Then (3.9) will be satisfied if we have

$$\rho < d^2 + \tfrac{1}{4} = (\tfrac{1}{8}\rho^4)^{2/5} + \tfrac{1}{4}$$

i.e. if $$g(\rho) = \frac{(4\rho - 1)^5}{\rho^8} < 16.$$

Since $g(\rho)$ is a decreasing function of ρ for $\rho \geqslant 4$, we have

$$g(\rho) \leqslant g(4) = \frac{15^5}{4^8} < 16.$$

Then the result follows from Lemma 8.

Proof of Theorem A (Continued)

From now on we can suppose that $\frac{1}{2} < d \leqslant 3$ and $\varphi(y, z, t, u)$ is not equivalent to $\rho\varphi_2$ or $\rho\varphi_3$, $\rho > 0$. By Lemma 5 we can find integers y_2, z_2, t_2, u_2 such that

$$0 < a = \varphi(y_2, z_2, t_2, u_2) \leqslant \left(\frac{2048}{729} \mid D \mid\right)^{1/4} \frac{8}{3\sqrt{3}}\, d^{5/4}. \qquad \ldots(3.12)$$

Further we can suppose that g.c.d. $(y_2, z_2, t_2, u_2) = 1$.

By a suitable unimodular transformation we can suppose that

$$\varphi(1, 0, 0, 0) = a$$

and write

$$\varphi(y, z, t, u) = a\{(y + fz + f't + f''u)^2 + \psi(z, t, u)\} \qquad \ldots(3.13)$$

where

$$|f| \leqslant \tfrac{1}{2}, \; |f'| \leqslant \tfrac{1}{2}, \; |f''| \leqslant \tfrac{1}{2}$$

and $\psi(z, t, u)$ is an indefinite quadratic form of the type (2, 1) and of determinant $|D|/a^4$.

Let $n < 2d \leqslant n + 1$, $n = 1, 2, ..., 5$.

Then in view of Lemma 8, it is enough to prove that there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ (mod 1) such that

$$-\frac{n^2 + 4d}{4a} < (y + fz + f't + f''u)^2 + \psi(z, t, u) \leqslant \frac{4d - 1}{4a} \qquad ...(3.14)$$

and to determine the cases when equality is necessary.

Lemma 12 — Let $\mu_1 = \dfrac{4d - 1 - a}{4a}$, $\lambda = \dfrac{8d + n^2 - 1}{4a}$. Let $\mu_2 > 0$ be a real number satisfying

$$\mu_2 \leqslant \begin{cases} \left(\dfrac{\lambda - 1}{2}\right)^2 + \dfrac{n^2 + 4d}{4a} & \text{if } \lambda \text{ is an integer} \\ \left(\dfrac{[\lambda]}{2}\right)^2 + \dfrac{n^2 + 4d}{4a} & \text{if } \lambda \text{ is not an integer.} \end{cases} \qquad ...(3.15)$$

Suppose that there exist $(z, t, u) \equiv (z_0, t_0, u_0)$ (mod 1) satisfying

$$-\mu_2 < \psi(z, t, u) \leqslant \mu_1. \qquad ...(3.16)$$

Then we can find $y \equiv y_0$ (mod 1) such that (3.14) holds. Further strict inequality in (3.16) implies strict inequality in (3.14).

PROOF : Using (3.12), one can easily verify that $\mu_1 > 0$ and $\lambda > 1$.

If $$-\frac{n^2 - 4d}{4a} < \psi(z, t, u) \leqslant \mu_1,$$

then choose $y \equiv y_0$ (mod 1) such that $|y + fz + f't + f''u| \leqslant \frac{1}{2}$. So that

$$-\frac{n^2 + 4d}{4a} < (y + fz + f't + f''u)^2 + \psi(z, t, u) \leqslant \tfrac{1}{4} + \mu_1 = \frac{4d - 1}{4a}$$

proving (3.14). Further strict inequality in (3.14) holds if we have strict inequality in (3.16).

Let now $$-\mu_2 < \psi(z, t, u) \leqslant -\frac{n^2 + 4d}{4a}.$$

Then $$0 \leqslant \beta^2 = -\frac{n^2 + 4d}{4a} - \psi(z, t, u) < -\frac{n^2 + 4d}{4a} + \mu_2.$$

Let $\alpha = fz + f't + f''u$; so that

$$(y + fz + f't + f''u)^2 + \psi(z, t, u) + \frac{n^2 + 4d}{4a} = (y + \alpha)^2 - \beta^2.$$

Then the result follows from Lemma 6 with d replaced by

$$\frac{4d - 1}{4a} + \frac{n^2 + 4d}{4a} = \frac{8d + n^2 - 1}{4a} = \lambda\,(> 1).$$

Lemma 13 — If $\lambda = \dfrac{8d + n^2 - 1}{4a} > 2$, then (3.14) and hence (3.7) is true with strict inequality.

PROOF : The result will follow from Lemma 12, if we prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\left\{\frac{(\lambda - 1)^2}{2} + \frac{n^2 + 4d}{4a}\right\} < \psi(z, t, u) < \frac{4d - 1 - a}{4a}. \quad \ldots(3.17)$$

Let

$$\nu = \left(\frac{\lambda - 1}{2}\right)^2 + \frac{n^2 + 4d}{4a} \Big/ \frac{4d - 1 - a}{4a}$$

$$= \frac{a(\lambda - 1)^2 + n^2 + 4d}{4d - 1 - a}. \quad \ldots(3.18)$$

Then $0 < \nu < \infty$, since $4d - 1 - a > 0$.

Also $\nu \geqslant 3$ if and only if $8d \leqslant n^2 + 3 + 3a + a(\lambda - 1)^2$

$$= n^2 + 3 + 3a + \frac{(8d + n^2 - 1 - 4a)^2}{16a}.$$

One can verify that this holds for $a < \dfrac{8d + n^2 - 1}{8}$, $\frac{1}{2} < d \leqslant 3$. Hence $3 \leqslant \nu < \infty$. Then by Lemma 7, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\nu\left\{\frac{1}{(1 + \nu)^2} \cdot \frac{|D|}{a^4}\right\}^{1/3} < \psi(z, t, u) \leqslant \left\{\frac{1}{(1 + \nu)^2} \cdot \frac{|D|}{a^4}\right\}^{1/3}$$

Therefore (3.17) will be satisfied if we have

$$\left\{\frac{1}{(1 + \nu)^2}\,\frac{|D|}{a^4}\right\}^{1/3} < \frac{4d - 1 - a}{4a}.$$

Substituting for ν from (3.18) and simplifying we see that this is so if

$$4^3.2.d^5 < a(4d - 1 - a)\{n^2 + 8d - 1 - a + a(\lambda - 1)^2\} = g(a) \text{ (say)}. \quad \ldots(3.19)$$

One can verify that $g'(a) < 0$ for $a < \dfrac{8d + n^2 - 1}{8}$. So $g(a)$ is a strictly decreasing function of a. Therefore

$$g(a) > g\left(\frac{8d + n^2 - 1}{8}\right) = \frac{1}{8^2}(8d + n^2 - 1)^3 (24d - 7 - n^2).$$

Now (3.19) will be satisfied with strict inequality if we have

$$f(d) = \frac{1}{d^5}(8d + n^2 - 1)^3 (24d - 7 - n^2) \geqslant 4^6.2. \qquad ...(3.20)$$

If $n = 1$, then $\frac{1}{2} < d \leqslant 1$ and $f(d) = \dfrac{8^3(24d - 8)}{d^2}$ increases for $d < \frac{2}{3}$ and decreases for $d > \frac{2}{3}$. Therefore

$$f(d) \geqslant \min\left(f(\tfrac{1}{2}), f(1)\right) = 4^6.2.$$

If $n \geqslant 2$, then $\dfrac{n}{2} < d \leqslant \dfrac{n+1}{2}$. One can easily see that $f(d)$ is a decreasing function of d. Therefore

$$f(d) \geqslant f\left(\frac{n+1}{2}\right) = \frac{2^5}{(n+1)^2}(n+3)^3 (12n + 5 - n^2)$$

$$\geqslant \frac{2^5}{3^2} 5^3 . 25 \ \ (\text{for } n \geqslant 2)$$

$$> 4^6.2.$$

Therefore (3.20) is true and hence (3.17) holds with strict inequality. (We notice that equality in (3.20) occurs only for $n = 1$ and $d = 1$. We need this in the next lemma.)

Lemma 14 — If $1 < \lambda \leqslant 2$, then again (3.14) and hence (3.7) is true.

PROOF : In view of Lemma 12, it is enough to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\left\{\frac{1}{4} + \frac{n^2 + 4d}{4a}\right\} < \psi(z, t, u) \leqslant \frac{4d - 1 - a}{4a}$$

i.e.
$$-\frac{n^2 + 4d + a}{4a} < \psi(z, t, u) \leqslant \frac{4d - 1 - a}{4a}. \qquad ...(3.21)$$

Let
$$\nu = \frac{n^2 + 4d + a}{4d - 1 - a}. \qquad ...(3.22)$$

Then $0 < \nu < \infty$ since $4d - 1 - a > 0$.

We prove that $\nu \geqslant 3$. This is so if and only if

$$n^2 + 3 + 4a \geqslant 8d. \qquad ...(3.23)$$

Since $a \geqslant \frac{8d + n^2 - 1}{8}$, we have

$$n^2 + 3 + 4a \geqslant n^2 + 3 + \frac{8d + n^2 - 1}{2} = \tfrac{3}{2} n^2 + \tfrac{5}{2} + 4d.$$

Therefore (3.23) will be satisfied if

$$\tfrac{3}{2} n^2 + \ +\tfrac{5}{2}\, 4d \geqslant 8d$$

i.e. if $\quad \tfrac{3}{2} n^2 + \tfrac{5}{2} \geqslant 4d.$

Since $d \leqslant \frac{n+1}{2}$, we see that this is true for $n \geqslant 1$. And $\nu = 3$ if and only if

$$n = 1,\, d = \frac{n+1}{2} = 1,\, a = \frac{8d + n^2 - 1}{8} = 1. \qquad \ldots(3.24)$$

Hence $3 \leqslant \nu < \infty$. By Lemma 7, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\nu \left\{\frac{1}{(1+\nu)^2} \; \frac{|D|}{a^4}\right\}^{1/3} < \psi(z, t, u) \leqslant \left\{\frac{1}{(1+\nu)^2} \; \frac{|D|}{a^4}\right\}^{1/3}$$

So (3.21) will be satisfied if we have

$$\left\{\frac{1}{(1+\nu)^2} \cdot \frac{|D|}{a^4}\right\}^{1/3} \leqslant \frac{4d - 1 - a}{4a}.$$

Substituting for ν from (3.22) and simplifying we see that this is so if

$$4^3.2.d^5 \leqslant a(4d - 1 - a)\,(n^2 + 8d - 1)^2 = g(a) \text{ (say)} \qquad \ldots(3.25)$$

$$g'(a) = (n^2 + 8d - 1)^2\,(4d - 1 - 2a).$$

Therefore $g(a)$ increases for $a \leqslant \frac{4d-1}{2}$ and decreases for $a \geqslant \frac{4d-1}{2}$. Since $\frac{8d + n^2 - 1}{8} < a \leqslant \frac{8}{3\sqrt{3}}\, d^{5/4}$ and $\frac{4d-1}{2} > \frac{8d + n^2 - 1}{8}$ for $n = 1, 2, 3$ but is $< \frac{8d + n^2 - 1}{8}$ for $n = 4, 5$, (3.25) will hold if we have

$$g\left(\frac{8}{3\sqrt{3}}\, d^{5/4}\right) \geqslant 4^3.2.d^5 \qquad \ldots(3.26)$$

and

$$g\left(\frac{8d + n^2 - 1}{8}\right) \geqslant 4^3.2.d^5. \qquad \ldots(3.27)$$

Now $g\left(\frac{8}{3\sqrt{3}}\, d^{5/4}\right) = \frac{8}{3\sqrt{3}}\, d^{5/4}\left(4d - 1 - \frac{8}{3\sqrt{3}}\, d^{5/4}\right)(n^2 + 8d - 1)^2.$

Therefore (3.26) will be satisfied if we have

$$f(d) = d^{-15/4}\left(4d - 1 - \frac{8}{3\sqrt{3}}\, d^{5/4}\right)(n^2 + 8d - 1)^2 > 48\sqrt{3}. \quad ...(3.28)$$

If $n = 1$, so that $(1/16)^{1/5} \leqslant d \leqslant 1$ and

$$f(d) = 8^2 . d^{-7/4}\left(4d - 1 - \frac{8}{3\sqrt{3}}\, d^{5/4}\right).$$

One can prove that there is a real number α, $7/12 < \alpha < 1$, such that $f(d)$ increases in $[(1/16)^{1/5}, \alpha]$ and decreases in $[\alpha, 1]$. Therefore

$$f(d) \geqslant \min\left(f\left(\left(\frac{1}{16}\right)^{1/5}\right), f(1)\right) > 48\sqrt{3}.$$

Thus (3.28) is true in this case.

Let now $n \geqslant 2$. Then $1 \leqslant n/2 < d \leqslant (n + 1)/2$.

In this case $f(d)$ is a decreasing function of d as the factors $\left(\frac{n^2 + 8d - 1}{d}\right)^2$ and $d^{-7/4}\left(4d - 1 - \frac{8}{3\sqrt{3}}\, d^{5/4}\right)$ are decreasing for $\frac{1}{2} < d \leqslant 3$. Therefore

$$\begin{aligned} f(d) &\geqslant f\left(\frac{n+1}{2}\right) \\ &= \frac{2^{15/4}}{(n+1)^{7/4}}\left\{2n + 1 - \frac{8}{3\sqrt{3}}\left(\frac{n+1}{2}\right)^{5/4}\right\}.(n+3)^2 \\ &\geqslant \frac{2^{15/4}}{3^{7/4}}\left\{5 - \frac{8}{3\sqrt{3}}\,(3/2)^{5/4}\right\}.25 \\ &\geqslant \frac{2^{15/4}}{3^{7/4}}.2.25 > 48\sqrt{3}. \end{aligned}$$

Hence (3.28) is satisfied in this case also. This proves (3.26) with strict inequality.

Now we prove (3.27).

$$g\left(\frac{8d + n^2 - 1}{8}\right) = \frac{1}{8^2}\,(n^2 + 8d - 1)^3\,(24d - 7 - n^2).$$

This will be $\geqslant 4^3.2.d^5$ if

$$\frac{1}{d^5}\,(8d + n^2 - 1)^3\,(24d - 7 - n^2) \geqslant 4^6.2.$$

This is the inequality (3.20) of Lemma 13. We verified this for $\frac{1}{2} < d \leqslant 3$ and $\frac{n}{2} < d \leqslant \frac{n+1}{2}$. Equality is necessary only if $n = 1$, $d = 1$. This proves (3.27) and hence (3.21).

4. The Case of Equality

Lemma 15 — Equality in (3.7) is needed if and only if Q is equivalent to the form Q_1.

Proof : Equality in (3.7) can occur only if we have equality in (3.14). From Lemma 12 and Lemma 14, this can be so only if $n = 1$, $d = 1$, $a = \dfrac{8d + n^2 - 1}{8} = 1$. But then from (3.24), $\nu = 3$.

Again by Lemma 7, for equality to occur, we must have

$$\psi(z, t, u) \sim \rho(z^2 + tu), (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, 0, 0) \pmod 1$$

or
$$\psi(z, t, u) \sim \rho(2z^2 + t^2 - u^2), (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1$$

where $\rho > 0$.

Case (*i*)

$$\psi(z, t, u) = \rho(z^2 + tu), (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, 0, 0) \pmod 1.$$

Then

$$\tfrac{1}{4} \cdot \rho^3 = \frac{|D|}{a^4} = 2d^5 = 2$$

implies $\rho = 2$.

Therefore $\varphi(y, z, t, u) = (y + fz + f't + f''u)^2 + 2z^2 + 2tu$. Again for equality to occur in (3.14), the inequality

$$-\frac{5}{4} = -\left(\frac{n^2}{4} + d\right) < F(y, z, t, u)$$
$$= (y + y_0 + f(z + \tfrac{1}{2}) + f't + f''u)^2 + 2(z + \tfrac{1}{2})^2 + 2tu$$
$$< d - \tfrac{1}{4} = \tfrac{3}{4}$$

should have no solution in integers y, z, t, and u. Now

$$0 < F(y, 0, 0, 0) = \left(y + y_0 + \frac{f}{2}\right)^2 + \tfrac{1}{2} < \tfrac{3}{4}$$

is solvable for integer y unless

$$y_0 + \tfrac{1}{2}f \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(4.1)}$$

Similarly considering $F(y, -1, 0, 0)$, $F(y, 0, 1, 0)$, $F(y, 0, 0, 1)$, if equality is to occur, we must have

$$y_0 - \tfrac{1}{2}f \equiv \tfrac{1}{2} \pmod 1 \qquad \text{...(4.2)}$$

$$y_0 + \tfrac{1}{2}f + f' \equiv \tfrac{1}{2} \pmod 1 \qquad \text{...(4.3)}$$

$$y_0 + \frac{f}{2} + f'' \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(4.4)}$$

From (4.1), (4.2), (4.3) and (4.4) we have

$$f \equiv f' \equiv f'' \equiv 0 \pmod 1.$$

Further since $|f| \leqslant \frac{1}{2}$, $|f'| \leqslant \frac{1}{2}$, $|f''| \leqslant \frac{1}{2}$ from (3.13), we must have

$$f = f' = f'' = 0,\ y_0 \equiv \tfrac{1}{2} \pmod 1.$$

Hence $\varphi(y, z, t, u) = y^2 + 2z^2 + 2tu$, $(y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, 0, 0) \pmod 1$

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + y^2 + 2z^2 + 2tu.$$

We first assert that $g = h' = g' = 0$.

If $g' \neq 0$, then using (3.4) we have

$$0 < Q(0, 0, 0, 0, 1) = g'^2 \leqslant \tfrac{1}{4} < 1 - \epsilon.$$

This contradicts (3.5). Therefore $g' = 0$. Similarly considering $Q(0, 0, 0, 1, 0)$ and $Q(0, 0, 1, -1, 1)$ we get that $h' = g' = 0$.

Therefore $Q(x, y, z, t, u) = (x + hy)^2 + y^2 + 2z^2 + 2tu$.

If equality is to occur in (3.7), the inequality

$$-1 = -d < G(x, y, z, t, u) = (x + x_0 + h(y + \tfrac{1}{2}))^2 + (y + \tfrac{1}{2})^2 + 2(z + \tfrac{1}{2})^2 + 2tu < d = 1$$

should have no solution in integers x, y, z, t, u.

Take $y = z = t = u = 0$ and choose integer x such that

$$\left| x + x_0 + \frac{h}{2} \right| \leqslant \tfrac{1}{2}.$$

so that $0 < G(x, 0, 0, 0, 0) = \left(x + x_0 + \frac{h}{2}\right)^2 + \frac{1}{4} + \frac{2}{4} \leqslant 1.$

Therefore for equality to occur we must have

$$x_0 + \frac{h}{2} \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(4.5)}$$

Similarly considering $G(x, -1, 0, 0, 0)$, for equality to occur, we must have

$$x_0 - \frac{h}{2} \equiv \tfrac{1}{2} \pmod 1. \qquad \text{...(4.6)}$$

From (4.5) and (4.6) we have

$$h \equiv 0 \pmod 1,$$

Since from (3.4), $|h| \leqslant \frac{1}{2}$, we must have $h = 0$, and $x_0 \equiv \frac{1}{2} \pmod 1$.

Therefore $Q(x, y, z, t, u) = x^2 + y^2 + 2z^2 + 2tu = Q_1$ and

$$(x_0, y_0, z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \pmod 1.$$

Considering congruence modulo 8, one can see that equality does occur in this case.

Case (*ii*)

$$\psi(z, t, u) = \rho(2z^2 + t^2 - u^2), (z_0, t_0, u_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1.$$

Proceeding as above, it is easy to prove that equality is necessary in (3.7) if and only if $Q = x^2 + y^2 + 2z^2 + t^2 - u^2$ and $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.

Now

$$\begin{aligned} Q &= x^2 + y^2 + 2z^2 + t^2 - u^2 \\ &= (x - t - u)^2 + y^2 + 2z^2 + 2(x - u)(t + u), \end{aligned}$$

Therefore Q is equivalent to Q_1 and we are reduced to case (i).

This completes the proof of Lemma 15.

The proof of Theorem A follows from Lemmas 8–15 and hence the theorem is proved.

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