Indian J. pure appl. Math., **11**(1) : 8–12, January 1980

SET-VALUED MAPPINGS ON BOUNDED METRIC SPACES

BRIAN FISHER

Department of Mathematics, The University, Leicester LE1 7RH, England

(*Received* 26 *March* 1979)

In this paper we consider mappings F, G of a complete, bounded metric space (X, d) into the set $B(X)$ of all nonempty subsets of X. For A, B in $B(X)$ we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}.$$

It is proved that if

$$\delta(Fx, Gy) \leqslant c \,.\, \max\{\delta(x, Fx),\ \delta(y, Gy),\ \delta(x, Gy),\ \delta(y, Fx),\ d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$, then there exists a unique point z in X such that $Fz = Gz = \{z\}$.

In the following, as in Fisher (1979a), we consider a mapping F of a complete metric space (X, d) into $B(X)$, the set of all nonempty, bounded subsets of X. If A is a nonempty subset of X we define the set FA by

$$FA = \bigcup_{a \in A} Fa$$

and we define the set $F^n A$ inductively by

$$F^n A = F(F^{n-1} A)$$

for $n = 2, 3, \ldots$. The function $\delta(A, B)$ with A, B in $B(X)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}.$$

In particular if A consists of a single point a we write

$$\delta(A, B) = \delta(a, B)$$

and if B also consists of a single point b we write

$$\delta(A, B) = \delta(a, b) = d(a, b).$$

It follows easily from the definition that

$$\delta(A, B) = \delta(B, A) \geqslant 0$$

and

$$\delta(A, B) \leqslant \delta(A, C) + \delta(C, B)$$

for all A, B, C in $B(X)$.

The following theorem has been proved by the author in one of his earlier papers (Fisher 1979a) :

Theorem 1 — Let F be a mapping of a complete metric space (X, d) into $B(X)$ satisfying the inequality

$$\delta(Fx, Fy) \leqslant c \,.\, \max\{\delta(x, Fx),\ \delta(y, Fy),\ \delta(x, Fy),\ \delta(y, Fx),\ d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$. If F also maps $B(X)$ into itself, that is $FA \in B(X)$ for $A \in B(X)$, then F has a unique fixed point z in X and further $Fz = \{z\}$.

In this theorem (see also Kaulgud and Pai 1975) the set-valued mapping F is defined to have a fixed point z if z is in Fz.

We now give a generalization of theorem 1 for two commuting mappings F, G of a complete metric space (X, d) into $B(X)$ in the special case where X is bounded so that in this case $B(X)$ will be the set of all nonempty subsets of X. Here we define the set-valued mappings F and G to commute if

$$F(Gx) = G(Fx)$$

for all x in X so that we will also have

$$F(GA) = G(FA)$$

for all A in X.

Theorem 2 — Let F, G be commuting mappings of a complete, bounded metric space (X, d) into $B(X)$ satisfying the inequality

$$\delta(Fx, Gy) \leqslant c \,.\, \max\{\delta(x, Fx),\ \delta(y, Gy),\ \delta(x, Gy),\ \delta(y, Fx),\ d(x, y)\} \quad \ldots(1)$$

for all x, y in X, where $0 \leqslant c < 1$. Then F and G have a common fixed point z. Further $Fz = Gz = \{z\}$ and z is the unique common fixed point of F and G.

PROOF : It follows from inequality (1) that if A, B are any sets in $B(X)$ then

$$\delta(Fa, Gb) \leqslant c \,.\, \max\{\delta(a, Fa),\ \delta(b, Gb),\ \delta(a, Gb),\ \delta(b, Fa),\ d(a, b)\}$$

for all a in A and b in B and so on taking the supremum over a in A and b in B of both sides of this inequality we have

$$\delta(FA, GB) \leqslant c \,.\, \max\{\delta(A, FA),\ \delta(B, GB),\ \delta(A, GB),\ \delta(B, FA),\ \delta(A, B)\}. \quad \ldots(2)$$

We note that since X is bounded

$$M = \sup\{\delta(A, B) : A, B \in B(X)\} < \infty.$$

We will now prove by induction that

$$\delta(F^nG^nA,\ F^nG^nB) \leqslant c^nM \quad \ldots(3)$$

for all A, B in $B(X)$ and $n = 1, 2, \ldots$. Since F and G commute we have

$$\begin{aligned}\delta(FGA, FGB) &= \delta(FGA, GFB) \\ &\leqslant c \,.\, \max\{\delta(GA, FGA), \delta(FB, GFB), \delta(GA, GFB), \delta(FB, FGA), \delta(GA, FB)\} \\ &\leqslant cM\end{aligned}$$

and so inequality (3) holds for all A, B in $B(X)$ when $n = 1$. Now assume that inequality (3) holds for all A, B in $B(X)$ and some n. Then

$$\begin{aligned}\delta(F^{n+1}G^{n+1}A, F^{n+1}G^{n+1}B) &= \delta(F^{n+1}G^{n+1}A, G^{n+1}F^{n+1}B) \\ &\leqslant c \,.\, \max\{\delta(F^nG^{n+1}A, F^{n+1}G^{n+1}A), \delta(G^nF^{n+1}B, G^{n+1}F^{n+1}B), \\ &\qquad \delta(F^nG^{n+1}A, G^{n+1}F^{n+1}B), \delta(G^nF^{n+1}B, F^{n+1}G^{n+1}A), \\ &\qquad \delta(F^nG^{n+1}A, G^nF^{n+1}B)\} \\ &\leqslant c^{n+1}M\end{aligned}$$

by our assumption. Inequality (3) now follows by induction.

For arbitrary $\epsilon > 0$ choose an integer N such that

$$c^N M < \epsilon.$$

It follows from inequality (3) that

$$\delta(F^nG^nA, F^mG^mB) < \epsilon \qquad \ldots(4)$$

for $m, n \geqslant N$ and all A, B in $B(X)$.

With a fixed A in $B(X)$ choose a point x_{2n} in F^nG^nA and a point x_{2n+1} in $F^nG^{n+1}A$ for $n = 1, 2, \ldots$. Then

$$d(x_{2n}, x_{2n+1}) \leqslant \delta(F^nG^nA, F^nG^n(GA)) \leqslant c^nM$$

by inequality (3) for $n = 1, 2, \ldots$. Similarly we have

$$d(x_{2n}, x_{2n+1}) \leqslant c^nM$$

for $n = 1, 2, \ldots$ and since $c < 1$, it follows that $\{x_n\}$ is a Cauchy sequence with a limit z in the complete metric space X. Then

$$\begin{aligned}\delta(z, F^nG^nA) &\leqslant d(z, x_{2m}) + \delta(x_{2m}, F^nG^nA) \\ &\leqslant d(z, x_{2m}) + \delta(F^mG^mA, F^nG^nA) \\ &< d(z, x_{2m}) + \epsilon\end{aligned}$$

by inequality (4) for $m, n \geqslant N$. Letting m tend to infinity we have

$$\delta(z, F^nG^nA) \leqslant \epsilon \qquad \ldots(5)$$

for $n \geqslant N$. Similarly

$$\begin{aligned}\delta(z, F^{n-1}G^nA) &\leqslant d(z, x_{2m}) + \delta(x_{2m}, F^{n-1}G^nA)\\ &\leqslant d(z, x_{2m}) + \delta(F^mG^mA, F^{n-1}G^{n-1}(GA))\\ &< d(z, x_{2m}) + \epsilon\end{aligned}$$

by inequality (3) for $m, n-1 \geqslant N$. Letting m tend to infinity we have

$$\delta(z, F^{n-1}G^nA) \leqslant \epsilon \qquad \ldots(6)$$

for $n > N$. Thus

$$\begin{aligned}\delta(F^{n-1}G^nA, Gz) &\leqslant \delta(F^{n-1}G^nA, z) + \delta(z, Gz)\\ &\leqslant \epsilon + \delta(z, Gz) \qquad \ldots(7)\end{aligned}$$

for $n > N$. On using inequality (2) we now have

$$\begin{aligned}\delta(F^nG^nA, Gz) &\leqslant c \, . \max\{\delta(F^{n-1}G^{n-1}(GA), F^nG^nA), \delta(z, Gz), \delta(F^{n-1}G^nA, Gz),\\ &\qquad \delta(z, F^nG^nA), \delta(F^{n-1}G^nA, z)\}\\ &\leqslant c \, . \max\{\epsilon, \delta(z, Gz), \epsilon + \delta(z, Gz), \epsilon, \epsilon\}\\ &= c\,[\epsilon + \delta(z, Gz)]\end{aligned}$$

by inequalities (4), (5), (6) and (7) for $n > N$. Since x_{2n} is in F^nG^nA, it follows that with $n > N$

$$\delta(x_{2n}, Gz) \leqslant \delta(F^nG^nA, Gz) \leqslant c\,[\epsilon + \delta(z, Gz)]$$

and letting n tend to infinity we have

$$\delta(z, Gz) \leqslant c\,[\epsilon + \delta(z, Gz)].$$

Since $c < 1$ and ϵ is arbitrary

$$\delta(z, Gz) = 0$$

and so

$$Gz = \{z\}.$$

Similarly we can prove that

$$Fz' = \{z'\}$$

for some z' in X. Then

$$d(z', z) = \delta(Fz', Gz) \leqslant cd(z', z)$$

by inequality (1). Since $c < 1$, it follows that $z = z'$ is a common fixed point of F and G.

Now suppose that F and G have a second fixed point w. Then

$$\delta(Fw, Gw) \leqslant c \, . \max\{\delta(w, Fw), \delta(w, Gw)\} \leqslant c\delta(Fw, Gw).$$

Since $c < 1$, it follows that

$$\delta(Fw, Gw) = 0$$

and so

$$Fw = Gw = \{w\}.$$

Thus

$$d(z, w) = \delta(Fz, Gw) \leqslant cd(z, w)$$

by inequality (1) and so z is the unique common fixed point of F and G. This completes the proof of the theorem.

Corollary — Let S, T be commuting mappings of a complete, bounded metric space (X, d) into itself satisfying the inequality

$$d(Sx, Ty) \leqslant c \,.\, \max\,\{d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sy), d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$. Then S and T have a common fixed point z. Further, z is the unique fixed point of S and T.

PROOF : Define mappings F and G of X into $B(X)$ by

$$Fx = \{Sx\},\ Gx = \{Tx\}$$

for all x in X. It follows that F and G satisfy inequality (1) for all x, y in X. Thus there exists a unique point z in X with

$$Fz = \{z\} = Gz$$

and so

$$Sz = z = Tz.$$

The result now follows.

The result of this corollary was given in Fisher (1979b). It was also shown that this result did not hold without the condition that S and T commute, although it is an open question whether the condition that X be bounded is necessary. It follows that the condition that F and G commute in Theorem 2 cannot be ommitted although it may be possible to omit the condition that X be bounded. Theorem 1 shows that in the particular case when $F = G$, the condition that X be bounded is not necessary.

REFERENCES

Fisher, B. (1979a). Set-valued mappings on metric spaces. *Fund. Math.*, (to appear).

———— (1979b). Results on common fixed points on bounded metric spaces. *Math. Sem. Notes Kobe Univ.*, 7, 73-80.

Kaulgud, N. N., and Pai, D. V. (1975). Fixed point theorems for set-valued mappings. *Nieuw Arch. Wisk.*, 23, 49-66.