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A NEW CLASS OF HYPERGEOMETRIC INTEGRAL EQUATIONS

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The purpose of this paper is to solve the Fredholm integral equation

$$\int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x)\, f(t)\, dt = g(x)$$

involving the hypergeometric function in the kernel. Riemann-Liouville fractional integration operator as well as Weyl operator is employed to reduce the left-hand side to a generalized Stieltjes transform which is then inverted to obtain the solution of the integral equation.

1. INTRODUCTION

In recent years many authors have solved Volterra integral equation of the first kind (Higgins 1964, Wimp 1965, Love 1967a, b, Sneddon 1968, Prabhakar 1972)

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} F\left(a, b; c; 1-\frac{t}{x}\right) f(t)\, dt = g(x) \qquad \ldots(1.1)$$

or some related integral equations with x to β as the range of integration, where the function $F(a, b; c; z)$ in the kernel is the ordinary hypergeometric function. Recently Love (1975) has discussed the Fredholm equation

$$\int_0^\infty \frac{t^{-b}}{\Gamma(b)} F(a, b; c; -x/t)\, f(t)\, dt = g(x), \quad 0 < x < \infty. \qquad \ldots(1.2)$$

He has used Riemann-Liouville fractional integrals to reduce the left side of (1.2) to a generalized Stieltjes transform and then to Stieltjes transform; the latter is finally inverted to obtain the solution of (1.2). In this paper we solve

$$\int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x)\, f(t)\, dt = g(x), \quad 0 < x < \infty \qquad \ldots(1.3)$$

which is a Fredholm integral equation of the first kind and is evidently more closely related to (1.1) than (1.2). Fractional integration has been employed to reduce the left side of (1.3) to a generalized Stieltjes transform; while doing so, we have used not only the Riemann-Liouville fractional integrals but also the Weyl fractional integrals. The solution is then obtained by inverting the generalized Stieltjes transform.

2. Definitions and Preliminary Results

Let $\mathcal{S}$ denote the space of functions f which are defined on $R^+ = [0, \infty)$ and satisfy

$$\text{(i)}\quad f \in C^\infty(R^+), \quad \text{(ii)}\quad \lim_{x \to \infty} [x^k f^{(r)}(x)] = 0$$

for all non-negative integers k and r. (iii) $f(x) = O(1)$ as $x \downarrow 0$. $\mathcal{S}$ corresponds to the space of good functions defined on the whole real line (see Miller 1975, Lighthill 1970).

For complex μ with Re $\mu > 0$ and locally integrable functions f, the Riemann-Liouville fractional integral is defined by

$$I^\mu f(x) = \int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)} f(t)\, dt.$$

For definitions of I^μ where Re $\mu \leqslant 0$ and other properties refer to Love (1967a), Prabhakar (1969, 1971). Also the Weyl fractional integral is defined by

$$J^\mu f(x) = \int_x^\infty \frac{(t-x)^{\mu-1}}{\Gamma(\mu)} f(t)\, dt, \text{ Re } \mu > 0$$

whenever the integral exists. J^μ possesses properties analogous to I^μ (see Love 1967b, Miller 1975).

For our discussion $\mathcal{S}$ is a sufficient class; in fact, restricting the discussion to $\mathcal{S}$ we are able to keep the details of the analysis involved to a bare minimum. For suitable $f \in \mathcal{S}$, it follows from Miller (1975) that $J^\mu f$ exists for all μ and is in $\mathcal{S}$. Moreover, it can be easily verified that $J^\mu J^\nu f = J^{\mu+\nu} f$ for all complex μ and ν. Indeed, on $\mathcal{S}$ the analogous results hold for I^μ.

3. Solution of (1.3)

We begin with the following Lemma.

Lemma — If a, b, c and k are complex parameters, $0 < \text{Re } c < \text{Re } k$, $x > 0$ and $t > 0$, then

$$\int_0^t \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} \frac{s^{c-1}}{\Gamma(c)} F\left(a, b; c; -\frac{s}{x}\right) ds$$

$$= \frac{t^{k-1}}{\Gamma(k)} F(a, b; k; -t/x). \qquad \text{...(3.1)}$$

PROOF : If z is any complex number such that $|z| > t$, then

$$\int_0^t \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} \frac{s^{c-1}}{\Gamma(c)} F\left(a, b; c; \frac{s}{z}\right) ds$$

$$= \int_0^t \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} \frac{s^{c-1}}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (s/z)^n}{(c)_n \, n!} ds$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (1/z)^n}{n!} \int_0^t \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} \frac{s^{c+n-1}}{\Gamma(c+n)} ds \qquad \ldots(3.2)$$

where the change in the order of integration and summation is justified by the dominated convergence theorem. Thus the integral (3.2) is equal to

$$\frac{t^{k-1}}{\Gamma(k)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (t/z)^n}{(k)_n \, n!} = \frac{t^{k-1}}{\Gamma(k)} F\left(a, b; k; \frac{t}{z}\right). \qquad \ldots(3.3)$$

The integral (3.2) and the function (3.3) are easily seen to be analytic functions of z in the complex plane cut along the line segment $[0, t]$. Putting $z = -x$, we get (3.1).

Theorem 1 — If a, b, c and k are complex parameters, Re $a > 0$, $0 <$ Re $c <$ Re k, $x > 0$ and $f \in \mathcal{S}$ then

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} F(a, b; k; -t/x) f(t) \, dt$$

$$= \int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x) J^{k-c} f(t) \, dt$$

PROOF : Using (3·1), we get

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} F\left(a, b; k; -\frac{t}{x}\right) f(t) \, dt$$

$$= \int_0^\infty f(t) \, dt \int_0^t \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} \frac{s^{c-1}}{\Gamma(c)} F\left(a, b; c; -\frac{s}{x}\right) ds$$

$$= \int_0^\infty \frac{s^{c-1}}{\Gamma(c)} F\left(a, b; c; -\frac{s}{x}\right) ds \int_0^\infty \frac{(t-s)^{k-c-1}}{\Gamma(k-c)} f(t)\, dt.$$

$$= \int_0^\infty \frac{s^{c-1}}{\Gamma(c)} F\left(a, b; c; -\frac{s}{x}\right) J^{k-c} f(s)\, ds.$$

The change in the order of integration is justified under the conditions stipulated in the theorem.

Theorem 2 — If a, b, c are complex parameters such that Re a, Re $c > 0$ and $f \in S$ then

$$\int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x) f(t)\, dt$$

$$= \frac{x^b}{\Gamma(a)} \int_0^\infty \frac{1}{(x+t)^b} t^{a-1} J^{c-a} f(t)\, dt.$$

PROOF : Substituting $J^{c-k}f$ for f in Theorem 1, we get

$$\int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x) f(t)\, dt$$

$$= \int_0^\infty \frac{t^{k-1}}{\Gamma(k)} F(a, b; k; -t/x) J^{c-k} f(t)\, dt.$$

Again replacing c by a and then f by $J^{c-k}f$ in Theorem 1, we get

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)} F(a, b; k; -t/x) J^{c-k} f(t)\, dt$$

$$= \frac{x^b}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}}{(x+t)^b} J^{c-a} f(t)\, dt.$$

Thus the theorem is proved.

Theorem 3 — If Re a, Re $c > 0$, Re $b > 1$ and $x^{-b}g(x) \in S$, then

$$\int_0^\infty \frac{t^{c-1}}{\Gamma(c)} F(a, b; c; -t/x) f(t)\, dt = g(x) \qquad ...(3.4)$$

has a solution $f \in \mathcal{S}$ given by

$$f(x) = \Gamma(a)\,\Gamma(b)\, J^{a-c} x^{1-a} I^{b-1} \lim_{n\to\infty} L_n\,[x^{-b} g(x)]. \qquad \ldots(3.5)$$

PROOF: Using Theorem 2, (3.4) can be written as

$$\frac{x^b}{\Gamma(a)} \int_0^\infty \frac{1}{(x+t)^b}\, t^{a-1} J^{c-a} f(t)\, dt = g(x) \qquad \ldots(3.6)$$

Now it follows from Lemma 5 of Love (1975) that if Re $\mu > 1$ and $f \in \mathcal{S}$ then

$$\int_0^\infty \frac{I^{1-\mu}\varphi(t)}{x+t}\, dt = \int_0^\infty \frac{\Gamma(\mu)}{(x+t)^\mu}\, \varphi(t)\, dt. \qquad \ldots(3.7)$$

In view of (3.7) eqn. (3.6) takes the form

$$\frac{x^b}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \frac{I^{1-b} t^{a-1} J^{c-a} f(t)}{x+t}\, dt = g(x). \qquad \ldots(3.8)$$

By an application of Theorem 9 of Widder (1946, p. 345) on the inversion of Stieltjes transform, we obtain

$$f(x) = \Gamma(a)\,\Gamma(b)\, J^{a-c} x^{1-a} I^{b-1} \lim_{n\to\infty} L_n[x^{-b} g(x)]$$

which is indeed a solution of (3.4), the operator L_n is defined by [Love 1975, (13); Widder 1946, p. 345]

$$L_n\,[\varphi(x)] = \frac{(-x)^{n-1}}{n!\,(n-2)!}\, \frac{d^{2n-1}}{dx^{2n-1}} \{x^n \varphi(x)\} \qquad \ldots(3.9)$$

where Widder's notation for $L_n\,[\varphi(x)]$ is $L_{n,x}\,[\varphi(t)]$.

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