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SINGULARLY PERTURBED INITIAL VALUE PROBLEMS FOR NONLINEAR DIFFERENTIAL SYSTEMS

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The asymptotic behaviour as $\varepsilon \to 0+$ of solutions of the initial value problem

$$d\bar{x}/dt = \bar{u}(t, \bar{x}, \bar{y}, \varepsilon), \qquad \bar{x}(a, \varepsilon) = \bar{A}(\varepsilon),$$

$$\varepsilon d\bar{y}/dt = \bar{v}(t, \bar{x}, \bar{y}, \varepsilon), \qquad \bar{y}(a, \varepsilon) = \bar{B}(\varepsilon), a < t \leqslant b,$$

where $\varepsilon > 0$ is a small parameter and $\bar{x}$, $\bar{u}$, $\bar{A}$ and $\bar{y}$, $\bar{v}$, $\bar{B}$ are N and M dimensional vector functions respectively, is established using the theory of first order differential inequalities. Under appropriate assumption, asymptotic estimates for solutions of the above initial value problem are constructed in terms of solutions of the unperturbed system

$$d\bar{x}/dt = \bar{u}(t, \bar{x}, \bar{y}, 0),\ 0 = \bar{v}(t, \bar{x}, \bar{y}, 0), \bar{x}(a, 0) = \bar{A}(0).$$

The estimates so obtained contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of solutions as a function of t and ε.

1. INTRODUCTION

The initial value problem (IVP)

$$\left.\begin{aligned} &\epsilon y'' + f(t, y, y', \epsilon) = 0, \ 0 \leqslant t \leqslant b < \infty \\ &y(0) = A(\epsilon), \ y'(0) = B(\epsilon) \end{aligned}\right\} \qquad \ldots(1.1)$$

where $\epsilon > 0$ is a small parameter, has been considered by Baxley (1974). Under suitable conditions on f which are quite different from the standard uniform Lipschitz condition, he obtained bounds on $y(t, \epsilon)$ and $y'(t, \epsilon)$ as $\epsilon \to 0+$. These bounds do not appear to contain the boundary layer term which is important for the analysis of the nonuniformity occurring in the derivative of a solution as $\epsilon \to 0+$. From the standard differential inequality theorems for first order differential equations (Coppel 1965), Howes (1976) deduced the existence and asymptotic behaviour of solutions of the following IVP :

$$\left.\begin{aligned} &a(t, \epsilon)\, y' = f(t, y, \epsilon), \ 0 < t \leqslant T \\ &y(0, \epsilon) = A \end{aligned}\right\} \qquad \ldots(1.2)$$

where $\epsilon > 0$ is a small parameter, $a(t, \epsilon) = a(t, 0) + O(\epsilon)$†, $a(t, \epsilon) > 0$, y and f are scalar functions. Weinstein and Smith (1975) also treated the IVP (1.1) in the case when f is linear with respect to y and y'. Using elementary comparison techniques, they studied the behaviour of solutions of the differential equation

$$\epsilon u'' + b(t, \epsilon) u' + c(t, \epsilon) u = f(t, \epsilon) \qquad ...(1.3)$$

for small values of the positive parameter ϵ and for values of t on a given interval $0 \leqslant t \leqslant t^*$. They obtained a few estimates for solutions of the eqn. (1.3) which are valid only when the coefficients $c(t, \epsilon)$ and $b(t, \epsilon)$ satisfy the overdamping condition :

$$4\epsilon c_1 < b_0^2 \qquad ...(1.4)$$

where

$$c(t, \epsilon) \leqslant c_1 \quad \text{and} \quad b(t, \epsilon) \geqslant b_0 > 0$$

for all $t \in [0, t^*]$ and sufficiently small $\epsilon > 0$.

However, it has been pointed out by Weinstein and Smith (1975) that the condition (1.4) can be removed by using either a theorem available in their article or the generalized maximum principle (O'Malley 1974). The present paper also considers the same eqn. (1.3) and obtains sharper estimates for its solution through an entirely different technique based on the theory of differential inequalities. The results thus obtained for (1.3) have been generalized to vector and higher order equations.

In section 2 we state two comparison theorems which shall be used in the rest of the paper. To make the paper self-contained, proofs are provided for the comparison theorems even though they are known to a certain extent (Adams and Spreuer 1975b, Adams 1976, Walter 1970). Section 3 deals with initial value problems for second order linear equations and obtains explicit bounds for its solutions. The results of section 3 are generalized to cover initial value problems for a system of nonlinear differential equations in section 4. As an application of the results obtained in section 4, IVPs for higher order equations are considered in section 5.

2. Preliminaries

Definition 2.1 — A vector function $\bar{f} = (f_1, ..., f_n)$ is called quasi-monotone increasing in $\bar{y} = (y_1, ..., y_n)$ if and only if every component f_i is monotone increasing in each of the variables y_j, $j \neq i$, $i, j = 1(1)\, n$, in the usual sense of monotone increasing.

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†Here and throughout the paper O denotes the standard Landau order symbol.

We now give two differential inequality theorems for a system of differential equations. The proof of the first theorem can be found in Walter (1970, p. 94). These theorems apply to the following IVP† :

$$\left.\begin{aligned} P_i\bar{y} &:= y_i' - f_i(t, y_1, \ldots, y_n) = 0, \quad a < t \leqslant b < \infty \\ R_i\bar{y} &:= y_i(a) = A_i, \quad i = 1(1)\, n \end{aligned}\right\} \quad \ldots(2.1)$$

where $\quad y_i \in Y := C^1(D) \cap C(\bar{D}),\ i = 1(1)\, n;\ \bar{D} := D \cup \partial D,$

$D := (a, b]$, and ∂D is the boundary of D; $\bar{y} = (y_1, \ldots, y_n)$; prime denotes differentiation with respect to t; $f_i \in C\,[D \times R^n, R]$.

Theorem 2.1 — Consider the IVP (2.1) and assume that $\bar{f} = (f_1, \ldots, f_n)$ is quasi-monotone increasing in $\bar{y} = (y_1, \ldots, y_n)$. Then we have the following implication for all $\bar{x} = (x_1, \ldots, x_n)$, $\bar{z} = (z_1, \ldots, z_n)$, $x_i, z_i \in Y$, $i = 1(1)\, n$:

$$\left.\begin{aligned} &P_i\bar{z} < P_i\bar{x} \text{ in } D,\ z_i(a) < x_i(a),\ i = 1(1)\, n \\ &\Rightarrow \\ &z_i(t) < x_i(t),\ t \in \bar{D},\ i = 1(1)\, n. \end{aligned}\right\} \quad \ldots(2.2)$$

Theorem 2.2 — Consider the IVP (2.1) and assume that $\bar{f}$ is quasi-monotone increasing in $\bar{y}$. Then for a solution $\bar{y} = (y_1, \ldots, y_n)$ of the IVP (2.1) the implication

$$\left.\begin{aligned} &P_i\bar{z} \leqslant P_i\bar{y} \leqslant P_i\bar{x},\ z_i(a) \leqslant y_i(a) \leqslant x_i(a),\ i = 1(1)\, n \\ &\Rightarrow \\ &z_i(t) \leqslant y_i(t) \leqslant x_i(t),\ t \in \bar{D},\ i = 1(1)\, n, \end{aligned}\right\} \quad \ldots(2.3)$$

is true for all $\bar{x} = (x_1, \ldots, x_n)$, $\bar{z} = (z_1, \ldots, z_n)$, $x_i, z_i \in Y$, $i = 1(1)\, n$, provided that each f_i, $i = 1(1)\, n$, satisfies the following conditions :

$$f_i(t, \bar{x} + \bar{s}) - f_i(t, \bar{x}) \leqslant l s_0 \quad \ldots(2.4)$$

and

$$f_i(t, \bar{z}) - f_i(t, \bar{z} - \bar{s}) \leqslant l s_0,\ i = 1(1)\, n \quad \ldots(2.5)$$

where $f_i(t, \bar{x}) := f_i(t, x_1, \ldots, x_n)$; $l \in R^+$; $\bar{s} = (s_1, \ldots, s_n)$, $s_i = s_0$, $i = 1(1)\, n$, $s_0 = s_0(t) > 0$ on $\bar{D}$.

†$A := B$ stands for A is defined by B.

The vector functions $\bar{x}$ and $\bar{z}$ are called the superfunction and subfunction respectively with respect to a solution $\bar{y}$ of the IVP (2.1).

PROOF : We give the proof for the superfunction. The subfunction case can be handled in a similar manner.

Let $\bar{s} = (s_1, \ldots, s_n)$ where

$$s_i(t) := C_1 \exp [t(l + C_2)], \; i = 1(1)\, n, \; C_1, C_2 \in R^+, \; t \in \bar{D}.$$

Then by using (2.3) and (2.4) we have

$$\begin{aligned} P_i [\bar{x} + \bar{s}] - P_i\bar{y} &\geqslant P_i [\bar{x} + \bar{s}] - P_i\bar{x} \\ &= s_i' - f_i(t, x_1 + s_1, \ldots, x_n + s_n) + f_i(t, x_1, \ldots, x_n) \\ &\geqslant s_i' - ls_i > 0, \; i = 1(1)\, n \end{aligned}$$

and

$$R_i [\bar{x} + \bar{s}] - R_i\bar{y} \geqslant R_i [\bar{x} + \bar{s}] - R_i\bar{x} = s_i(a) > 0, \; i = 1(1)\, n.$$

Therefore,

$$P_i [\bar{x} + \bar{s}] > P_i [\bar{y}], \; R_i [\bar{x} + \bar{s}] > R_i [\bar{y}], \; i = 1(1)\, n$$

which, by Theorem 2.1, yield

$$y_i(t) < x_i(t) + s_i(t), \; t \in \bar{D}, \; i = 1(1)\, n.$$

Since C_1 is arbitrary, we have

$$y_i(t) \leqslant x_i(t), \; t \in \bar{D}, \; i = 1(1)\, n.$$

3. Initial Value Problems for A Second Order Linear Equation

In this section we consider the following IVP (3.1) and obtain estimates for the solution and its derivative :

$$\left.\begin{aligned} &\epsilon y'' + b(t, \epsilon)\, y' + c(t, \epsilon)\, y = d(t, \epsilon), \; t \in D := (a, b] \\ &y(a, \epsilon) = A(\epsilon), \; y'(a, \epsilon) = B(\epsilon) \end{aligned}\right\} \qquad \ldots(3.1)$$

where $\epsilon > 0$ is a small parameter, b, c and d are assumed to be given continuous functions of two variables t and ϵ for $t \in \bar{D}$ and for all small values of ϵ;

$$A(\epsilon) = A(0) + O(\epsilon), \quad B(\epsilon) = B(0) + O(\epsilon).$$

We assume that there are constants b_0, c_0 and c_1 in terms of which b and c satisfy the uniform bounds

$$c_0 \leqslant c(t, \epsilon) \leqslant c_1 \qquad \ldots(3.2)$$

and

$$b(t, \epsilon) \geqslant b_0 > 0. \qquad \ldots(3.3)$$

Weinstein and Smith (1975) discussed fairly the IVP (3.1) under the assumption that the coefficients b and c satisfy the overdamping condition (1.4). In fact, they constructed bounds for the solution of the IVP (3.1) in the two cases $c(t, \epsilon) \leqslant 0$ and $c(t, \epsilon) > 0$. We, in the present paper, discuss the same IVP (3.1) using an approach which is different from that of Baxley (1974) and Weinstein and Smith (1975). First we discuss the case $c \leqslant 0$ and subsequently relax this condition.

The IVP (3.1) is now associated with a system which is similar to that of (2.1) so that Theorem 2.2 can be applied. In fact the IVP (3.1) is replaced by the system

$$\left.\begin{aligned} P_1\bar{y} &:= y_1' - y_2 = 0 \\ P_2\bar{y} &:= \epsilon y_2' + b(t, \epsilon)\, y_2 + c(t, \epsilon)\, y_1 = d(t, \epsilon),\ t \in D \\ R_1\bar{y} &:= y_1(a) = A(\epsilon) \\ R_2\bar{y} &:= y_2(a) = B(\epsilon) \end{aligned}\right\} \qquad \ldots(3.4)$$

where $\quad y_1 := y$ and $\bar{y} := (y_1, y_2)$.

If we assume the condition

$$c(t, \epsilon) \leqslant 0 \qquad \ldots(3.5)$$

then the system (3.4) is quasimonotone according to the Definition 2.1 and hence (since the system is linear) the implication (2.3) is valid for the IVP (3.4).

First we state and prove three theorems (Theorems 3.1 – 3.3) with (3.5) satisfied and subsequently relax the condition (3.5) in Theorem 3.4.

In the following analysis, M_i, $i = 1, 2, \ldots,$ are constants independent of ϵ.

The condition (3.5) has to be taken as one of the hypotheses of Theorems 3.1 – 3.3.

Theorem 3.1 — Consider the IVP (3.1) and assume that $A(\epsilon) \equiv 0 \equiv B(\epsilon)$. Then we have

$$|\, y(t, \epsilon)\, | \leqslant \|\, d\, \|\, e^{m(t-a)} \qquad \ldots(3.6)$$

$$|\, y'(t, \epsilon)\, | \leqslant m\, \|\, d\, \|\, e^{m(t-a)} \qquad \ldots(3.7)$$

where $m \in R^+$ with $mb_0 + c_0 \geqslant 1$ and

$$\|\, d\, \| = \sup_{t \in D}\ |\, d(t, \epsilon)\, |\, . \qquad \ldots(3.8)$$

PROOF : Consider the IVP (3.4) which is associated with the IVP (3.1). Because of the assumption on $c(t, \epsilon)$, the implication (2.3) is true for the system (3.4). The vector function $\bar{x} = (x_1, x_2)$ defined by

$$x_1(t, \epsilon) = \| d \| e^{m(t-a)}, \ t \in \bar{D}$$

$$x_2(t, \epsilon) = m \| d \| e^{m(t-a)}$$

is a superfunction with respect to a solution $\bar{y} = (y_1, y_2)$,

$y_1' = y_2, y_1 := y$, of the IVP (3.4). For,

$$P_1\bar{x} = x_1' - x_2 = 0 = P_1\bar{y},$$

$$P_2\bar{x} = x_1(t, \epsilon)\,[\epsilon m^2 + mb(t, \epsilon) + c(t, \epsilon)]$$

$$\geqslant x_1(t, \epsilon)\,[mb_0 + c_0] \geqslant \| d \| \geqslant d(t, \epsilon) = P_2\bar{y}$$

$$x_1(a, \epsilon) \geqslant 0 = y_1(a, \epsilon) \quad \text{and} \quad x_2(a, \epsilon) \geqslant 0 = y_2(a, \epsilon).$$

Hence, by Theorem 2.2, we have

$$y_i(t, \epsilon) \leqslant x_i(t, \epsilon), \ t \in \bar{D}, \ i = 1, 2.$$

Similarly

$$-x_i(t, \epsilon) \leqslant y_i(t, \epsilon), \ t \in \bar{D}, \ i = 1, 2.$$

Then the results (3.6) and (3.7) follow.

Theorem 3.2 — Consider the IVP (3.1) and assume that $d(t, \epsilon) \equiv 0 \equiv A(\epsilon)$. Then we have

$$| y(t, \epsilon) | \leqslant \epsilon \, | B(\epsilon) | \, M_1 \, [e^{m(t-a)} - e^{-b_0(t-a)/\epsilon}] \qquad \ldots(3.9)$$

$$| y'(t, \epsilon) | \leqslant \epsilon \, | B(\epsilon) | \, M_2 e^{m(t-a)} + [e^{-b_0(t-a)/\epsilon}] \, | B(\epsilon) | \qquad \ldots(3.10)$$

where $t \in \bar{D}; m \in R^+$ with $mb_0 + c_0 \geqslant 0$.

PROOF : Consider the IVP (3.4) which is associated with the IVP (3.1). Because of the assumption on $c(t, \epsilon)$, the implication (2.3) is true for the system (3.4). The vector function $\bar{x} = (x_1, x_2)$ defined by

$$x_1(t, \epsilon) = \epsilon \, | B(\epsilon) | \, [e^{m(t-a)} - e^{-b_0(t-a)/\epsilon}]/[m\epsilon + b_0]$$

$$x_2(t, \epsilon) = | B(\epsilon) | \, [m\epsilon e^{m(t-a)} + b_0 e^{-b_0(t-a)/\epsilon}]/[m\epsilon + b_0]$$

is the superfunction with respect to a solution $\bar{y} = (y_1, y_2)$,

$y_1' = y_2, y_1 := y$, of the IVP (3.4). For,

$$P_1\bar{x} = x_1' - x_2 = 0 = P_1\bar{y}$$

$$\begin{aligned} P_2\bar{x} &= [\epsilon/(m\epsilon + b_0)] \mid B(\epsilon) \mid e^{m(t-a)} [\epsilon m^2 + mb(t, \epsilon) + c(t, \epsilon)] \\ &\quad - [\mid B(\epsilon) \mid /(m\epsilon + b_0)] e^{-b_0(t-a)/\epsilon} [b_0^2 - b_0 b(t, \epsilon) + \epsilon c(t, \epsilon)] \\ &\geqslant 0 = P_2\bar{y}, \text{ [since } c(t, \epsilon) \leqslant 0 \quad \text{and} \quad mb_0 + c_0 \geqslant 0] \end{aligned}$$

$$R_1\bar{x} = x_1(a, \epsilon) = 0 = R_1\bar{y}, R_2\bar{x} = x_2(a, \epsilon) = \mid B(\epsilon) \mid \geqslant R_2\bar{y}.$$

Hence, by Theorem 2.2, we have

$$y_i(t, \epsilon) \leqslant x_i(t, \epsilon), \ t \in \bar{D}, \ i = 1, 2.$$

Similarly

$$- x_i(t, \epsilon) \leqslant y_i(t, \epsilon), \ t \in \bar{D}, \ i = 1, 2.$$

Therefore the results (3.9) and (3.10) follow.

Making use of the above theorems, we shall study the asymptotic behaviour of the IVP (3.1).

Theorem 3.3 — Consider the IVP (3.1). Then we have

$$\begin{aligned} \mid y(t, \epsilon) - u(t) \mid &\leqslant \epsilon M_3 e^{m(t-a)} + \gamma_1(\epsilon) \\ &\quad + \epsilon M_4 \mid B(\epsilon) - u'(a) \mid [e^{m(t-a)} - e^{-b_0(t-a)/\epsilon}] \end{aligned} \qquad \text{...(3.11)}$$

$$\begin{aligned} \mid y'(t, \epsilon) - u'(t) \mid &\leqslant \epsilon M_5 e^{m(t-a)} + \epsilon M_6 \mid B(\epsilon) - u'(a) \mid e^{m(t-a)} \\ &\quad + \mid B(\epsilon) - u'(a) \mid e^{-b_0(t-a)/\epsilon} + \gamma_2(\epsilon) \end{aligned} \qquad \text{...(3.12)}$$

where $t \in \bar{D}$, $\gamma_1(\epsilon)$ and $\gamma_2(\epsilon)$ approach zero as $\epsilon \to 0+$, and u is the solution of the IVP

$$\left.\begin{aligned} & b_0(t)\, u' + c_0(t)\, u = d_0(t), \ t \in D \\ & u(a) = A(0), \ b_0(t) = \operatorname*{Lt}_{\epsilon \to 0+} b(t, \epsilon) \ \text{ and etc.} \end{aligned}\right\} \qquad \text{...(3.13)}$$

Consequently,

$$\operatorname*{Lt}_{\epsilon \to 0+} y(t, \epsilon) = u(t), \quad a \leqslant t \leqslant b \qquad \text{...(3.14)}$$

and

$$\operatorname*{Lt}_{\epsilon \to 0+} y'(t, \epsilon) = u'(t), \ a < t \leqslant b. \qquad \text{...(3.15)}$$

PROOF : Let $w(t, \epsilon)$ be the solution of the IVP

$$\left.\begin{aligned} b(t, \epsilon)\, w' + c(t, \epsilon)\, w &= d(t, \epsilon), \ t \in D \\ w(a, \epsilon) &= A(\epsilon). \end{aligned}\right\} \qquad \text{...(3.16)}$$

By making appropriate assumptions on b, c and d, one can have

$$\| w''(t, \epsilon) \| \leqslant mM_3 \qquad \ldots(3.17)$$

where $\| \cdot \|$ is defined as in (3.8).

Furthermore, from a standard theorem on the continuity of solutions of IVPs (Coddington and Levinson 1955, p. 58) it follows that

$$\operatorname*{Lt}_{\epsilon \to 0+} w(t, \epsilon) = u(t), \; t \in \bar{D} \qquad \ldots(3.18)$$

where u is the solution of the IVP (3.13).

Let $v(t, \epsilon)$ be the solution of the IVP

$$\left.\begin{aligned} &\epsilon v'' + b(t, \epsilon)\, v' + c(t, \epsilon)\, v = d(t, \epsilon) + \epsilon w''(t, \epsilon), \; t \in D \\ &v(a, \epsilon) = A(\epsilon), \; v'(a, \epsilon) = B(\epsilon). \end{aligned}\right\} \qquad \ldots(3.19)$$

From Theorem 3.1, we have

$$| y(t, \epsilon) - v(t, \epsilon) | \leqslant \epsilon M_3 e^{m(t-a)} \qquad \ldots(3.20)$$

$$| y'(t, \epsilon) - v'(t, \epsilon) | \leqslant \epsilon M_5 e^{m(t-a)}. \qquad \ldots(3.21)$$

Also, from Theorem 3.2 it follows that

$$| v(t, \epsilon) - w(t, \epsilon) | \leqslant \epsilon M_4 \,| B(\epsilon) - w'(a, \epsilon) |\, [e^{m(t-a)} - e^{-b_0(t-a)/\epsilon}] \qquad \ldots(3.22)$$

$$\begin{aligned} | v'(t, \epsilon) - w'(t, \epsilon) | \leqslant{} & \epsilon M_6 \,| B(\epsilon) - w'(a, \epsilon) |\, e^{m(t-a)} \\ & + | B(\epsilon) - w'(a, \epsilon) |\, e^{-b_0(t-a)/\epsilon}. \end{aligned} \qquad \ldots(3.23)$$

Finally, we have

$$| y(t, \epsilon) - u(t) | \leqslant | y(t, \epsilon) - w(t, \epsilon) | + | w(t, \epsilon) - u(t) | \qquad \ldots(3.24)$$

and

$$| y'(t, \epsilon) - u'(t) | \leqslant | y'(t, \epsilon) - w'(t, \epsilon) | + | w'(t, \epsilon) - u'(t) | . \qquad \ldots(3.25)$$

The results (3.11) and (3.12) follow immediately from (3.20) - (3.25) and (3.18).

In Theorems 3.1 - 3.2 above we have assumed the validity of the condition (3.5) and discussed the asymptotic behaviour of solution of the IVP (3.1). When the condition (3.5) is not satisfied, i.e., the system (3.4) which is associated with (3.1) is not quasi-monotone, we can still discuss the asymptotic behaviour by defining a new quasi-monotone system, solution of which contains solution of the IVP (3.4) (Adams 1976, Adams and Spreuer 1975a, b). We set

$$\left.\begin{aligned} c^{+}(t,\epsilon) &:= c(t,\epsilon), \text{ if } c(t,\epsilon) \geqslant 0 \text{ and zero otherwise} \\ c^{-}(t,\epsilon) &:= c(t,\epsilon) - c^{+}(t,\epsilon), \end{aligned}\right\} \quad \ldots(3.26)$$

and consider the IVP defined by the system

$$\left.\begin{aligned} \hat{P}_1\hat{y} &:= \hat{y}_1' - \hat{y}_2 = 0,\ t \in D \\ \hat{P}_2\hat{y} &:= \epsilon\hat{y}_2' + b(t,\epsilon)\,\hat{y}_2 - c^{+}(t,\epsilon)\,\hat{y}_3 + c^{-}(t,\epsilon)\,\hat{y}_1 = -d(t,\epsilon) \\ \hat{P}_3\hat{y} &:= \hat{y}_3' - \hat{y}_4 = 0,\ t \in D \\ \hat{P}_4\hat{y} &:= \epsilon\hat{y}_4' + b(t,\epsilon)\,\hat{y}_4 - c^{+}(t,\epsilon)\,\hat{y}_1 + c^{-}(t,\epsilon)\,\hat{y}_3 = d(t,\epsilon) \\ \hat{R}_1\hat{y} &:= \hat{y}_1(a,\epsilon) = -A(\epsilon),\ \hat{R}_3\hat{y} := \hat{y}_3(a,\epsilon) = A(\epsilon) \\ \hat{R}_2\hat{y} &:= \hat{y}_2(a,\epsilon) = -B(\epsilon),\ \hat{R}_4\hat{y} := \hat{y}_4(a,\epsilon) = B(\epsilon) \end{aligned}\right\} \quad \ldots(3.27)$$

where $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4)$.

It is obvious that the system (3.27) is quasi-monotone according to the definition 2.1. Also if $\bar{y} = (y_1, y_2)$ is a solution of the IVP (3.4), then

$$\hat{y} = (-y_1, -y_2, y_1, y_2)$$

is a solution of the IVP (3.27). Hence if the IVP (3.27) has a unique solution, then it must be of the form $(-y_1, -y_2, y_1, y_2)$, where (y_1, y_2) is the solution of the IVP (3.4). This property has been used to prove the following theorem without assuming the condition (3.5).

Theorem 3.4 — Consider the IVP (3.1). Then the estimates (3.11) and (3.12) are valid.

PROOF: Consider IVP (3.4). If the system (3.4) is not quasi-monotone according to the Definition 2.1, i e., the condition (3.5) is not true, then introduce a quasi-monotone system (3.27). Following the same procedure as in the proof of Theorem 3.3, we have the following estimates for the system (3.27).

$$\begin{aligned} |\,\hat{y}_1(t,\epsilon) - \hat{u}_1(t)\,| \leqslant{}& \epsilon M_3 e^{m(t-a)} + \epsilon M_4\,|-B(\epsilon) - \hat{u}_1'(a)\,|\,[e^{m(t-a)} \\ &- e^{-b_0(t-a)/\epsilon}] + \gamma_1(\epsilon) \end{aligned} \quad \ldots(3.28)$$

$$\begin{aligned} |\,\hat{y}_2(t,\epsilon) - \hat{u}_2(t)\,| \leqslant{}& \epsilon M_5 e^{m(t-a)} + \epsilon M_6\,|-B(\epsilon) - \hat{u}_1'(a)\,|\,e^{m(t-a)} \\ &+ |-B(\epsilon) - \hat{u}_1'(a)\,|\,e^{-b_0(t-a)/\epsilon} + \gamma_2(\epsilon) \end{aligned} \quad \ldots(3.29)$$

$$\begin{aligned} |\,\hat{y}_3(t,\epsilon) - \hat{u}_3(t)\,| \leqslant{}& \epsilon M_3 e^{m(t-a)} + \epsilon M_4\,|\,B(\epsilon) - \hat{u}_3'(a)\,|\,[e^{m(t-a)} \\ &- e^{-b_0(t-a)/\epsilon}] + \gamma_1(\epsilon) \end{aligned} \quad \ldots(3.30)$$

$$| \hat{y}_4(t, \epsilon) - \hat{u}_4(t) | \leqslant \epsilon M_5 e^{m(t-a)} + \epsilon M_6 \, | B(\epsilon) - \hat{u}_3'(a) | \, e^{m(t-a)}$$
$$+ \, | B(\epsilon) - \hat{u}_3'(a) | \, e^{-b_0(t-a)/\epsilon} + \gamma_2(\epsilon) \qquad \text{...(3.31)}$$

where $\gamma_1(\epsilon)$ and $\gamma_2(\epsilon)$ go to zero as $\epsilon \to 0+$ and $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4)$ is the solution of the IVP

$$\left.\begin{array}{l} \hat{u}_1' - \hat{u}_2 = 0, \; b_0(t)\, \hat{u}_2 - c_0^+(t)\, \hat{u}_3 + c_0^-(t)\, \hat{u}_1 = -d_0 \\ \hat{u}_3' - \hat{u}_4 = 0, \; b_0(t)\, \hat{u}_4 + c_0^+(t)\, \hat{u}_1 + c_0^-(t)\, \hat{u}_3 = d_0 \\ \hat{u}_1(a) = -A(0), \; \hat{u}_3(a) = A(0). \end{array}\right\} \qquad \text{...(3.32)}$$

Since IVPs (3.27) and (3.32) are linear and $b(t, \epsilon)$ and $c(t, \epsilon)$ satisfy the conditions (3.2) and (3.3), they have unique solutions. Hence solutions of the IVPs (3.27) and (3.32) are respectively of the forms $(-y_1, -y_2, y_1, y_2)$ and $(-u_1, -u_2, u_1, u_2)$ where (y_1, y_2) and (u_1, u_2) are respectively solutions of the IVPs (3.4) and (3.13) with $u_1 := u, u_2 = u_1'$. Therefore the estimates (3.28) - (3.31) lead to the results (3.11) and (3.12).

4. Initial Value Problems for a System of Nonlinear Differential Equations

In Section 3, estimates are obtained for solutions of the IVPs for a scalar second order linear equation. Guided by this experience we, in the present section, derive estimates for solutions of the following IVPs :

$$\left.\begin{array}{l} P_i\bar{z} := x_i' - u_i(t, \bar{x}, \bar{y}, \epsilon) = 0, \; t \in D, \; i = 1(1)\, N \\ P_{N+j}\bar{z} := \epsilon y_j' - v_j(t, \bar{x}, \bar{y}, \epsilon) = 0, \; j = 1(1)\, M \\ R_i\bar{z} := x_i(a, \epsilon) = A_i(\epsilon) = A_i(0) + O(\epsilon) \\ R_{N+j}\bar{z} := y_j(a, \epsilon) = B_j(\epsilon) = B_j(0) + O(\epsilon) \end{array}\right\} \qquad \text{...(4.1)}$$

where $\bar{z} := (\bar{x}, \bar{y}) = (x_1, \ldots, x_N, y_1, \ldots, y_M)$.

Making use of the technique of matched asymptotic expansions, O'Malley (1968, 1974) has discussed the asymptotic behaviour of solutions of the IVP (4.1). Also Hoppensteadt (1971) and Vasiléva (1963), with the help of some other techniques, studied the behaviour of solutions of the IVP (4.1). With the aid of the experience gained on the IVPs for scalar second order linear equations and using Theorem 2.2, we obtain estimates for solutions of IVP (4.1). The estimates thus derived would contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of the solution of (4.1) in t and ϵ.

The degenerate problem for the IVP (4.1) is given by

$$\left.\begin{array}{l} x_i' - u_i(t, \bar{x}, \bar{y}, 0) = 0,\ t \in D, x_i(a) = A_i(0) \\ v_j(t, \bar{x}, \bar{y}, 0) = 0,\ i = 1(1)\, N,\ j = 1(1)\, M. \end{array}\right\} \qquad \ldots(4.2)$$

We shall make the following assumptions.

I. The vector function $\bar{w} = (\bar{u}, \bar{v})$, $\bar{u} = (u_1, \ldots, u_N)$, $\bar{v} = (v_1, \ldots, v_M)$, is quasi-monotone increasing in $\bar{z} = (\bar{x}, \bar{y})$, according to the definition 2.1,

II. There exists a vector function $\bar{\phi} = (\phi_1, \ldots, \phi_M)$ such that

$$v_j(t, \bar{x}, \bar{\phi}, 0) = 0,\ j = 1(1)\, M \qquad \ldots(4.3)$$

and the resulting IVP

$$x_i' - u_i(t, \bar{x}, \bar{\phi}, 0) = 0,\ x_i(a) = A_i(0),\ t \in D,\ i = 1(1)\, N \qquad \ldots(4.4)$$

has a unique solution $\bar{X} = (X_1, \ldots, X_N)$ on $\bar{D}$ such that

$$\left.\begin{array}{ll} & v_j(t, \bar{X}, \bar{Y} + \bar{\beta}, \epsilon) - v_j(t, \bar{X}, \bar{Y}, \epsilon) \leqslant -k\beta_0 \\ \text{where} & \bar{\beta} = (\beta_1, \ldots, \beta_M),\ \beta_j(t, \epsilon) = \beta_0(t, \epsilon) > 0 \text{ on } \bar{D} \\ & j = 1(1)\, M,\ k > 0,\ Y_i(t) = \phi_i(t, \bar{X}). \end{array}\right\} \qquad \ldots(4.5)$$

III. For some constant $l > 0$,

$$\left.\begin{array}{ll} & u_i(t, \bar{X} + \bar{\alpha}, \bar{Y} + \bar{\beta}, \epsilon) - u_i(t, \bar{X}, \bar{Y}, \epsilon) \leqslant l\,[\alpha_0 + \beta_0] \\ \text{where} & \bar{\alpha} = (\alpha_1, \ldots, \alpha_N),\ \alpha_i(t, \epsilon) = \alpha_0(t, \epsilon) > 0 \text{ on } \bar{D}, i = 1(1)\, N \end{array}\right\} \qquad \ldots(4.6)$$

$$v_j(t, \bar{X} + \bar{\alpha}, \bar{Y} + \bar{\beta}, \epsilon) - v_j(t, \bar{X}, \bar{Y} + \bar{\beta}, \epsilon) \leqslant l\alpha_0,\ j = 1(1)\, M. \qquad \ldots(4.7)$$

We now establish the asymptotic behaviour of a solution of the IVP (4.1).

Theorem 4.1 — Consider the IVP (4.1) and let $\bar{x} = (x_1, \ldots, x_N)$, $\bar{y} = (y_1, \ldots, y_M)$ and $\bar{X} = (X_1, \ldots, X_N)$, $\bar{Y} = (Y_1, \ldots, Y_M)$ be respectively solutions of the IVPs (4.1) and (4.2). Further assume that

$$u_i(t, \bar{X}, \bar{Y}, \epsilon) - u_i(t, \bar{X}, \bar{Y}, 0) = O(\epsilon),\ i = 1(1)\, N \qquad \ldots(4.8)$$

$$v_j(t, \bar{X}, \bar{Y}, \epsilon) = O(\epsilon),\ j = 1(1)\, M. \qquad \ldots(4.9)$$

Then under the above assumptions I – III we have

$$| x_i(t, \epsilon) - X_i(t) | \leqslant \epsilon M_3 e^{m(t-a)} + \epsilon M_4 [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] [e^{m(t-a)} - e^{-k(t-a)/\epsilon}], \; i = 1(1)\, N \qquad ...(4.10)$$

and

$$| y_j(t, \epsilon) - Y_j(t) | \leqslant \epsilon M_5 e^{m(t-a)} + \epsilon M_6 [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] \, e^{m(t-a)} + [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] \, e^{-k(t-a)/\epsilon},$$

$$t \in \bar{D}, \; j = 1(1)\, M, \; m > 0. \qquad ...(4.11)$$

PROOF : Because of the conditions I – III stated above, the implication (2.3) is true for the IVP (4.1). Also, from the hypotheses of the present theorem we have

$$| A_i(\epsilon) - A_i(0) | \leqslant \epsilon M_3 \qquad ...(4.12)$$

$$| u_i(t, \bar{X}, \bar{Y}, \epsilon) - u_i(t, \bar{X}, \bar{Y}, 0) | \leqslant \epsilon M_7, \; i = 1(1)\, N \qquad ...(4.13)$$

$$| v_j(t, \bar{X}, \bar{Y}, \epsilon) | \leqslant \epsilon M_8 \qquad ...(4.14)$$

and

$$| Y_j'(t) | \leqslant M_9, \; j = 1(1)\, M. \qquad ...(4.15)$$

Consider the vector function $\bar{z}^* = (\bar{x}^*, \bar{y}^*)$ defined be

$$\bar{x}^* = (x_1^*, ..., x_N^*) \quad \text{and} \quad \bar{y}^* = (y_1^*, ..., y_M^*)$$

$$x_i^* (t, \epsilon) = X_i(t) + \eta_0(t, \epsilon), \; i = 1(1)\, N, \; t \in \bar{D} \qquad ...(4.16)$$

$$y_j^* (t, \epsilon) = Y_j(t) + \gamma_0(t, \epsilon), \; j = 1(1)\, M, \; t \in \bar{D} \qquad ...(4.17)$$

where $\bar{\eta} = (\eta_1, ..., \eta_N), \; \eta_i = \eta_0, \; i = 1(1)\, N$

$$\bar{\gamma} = (\gamma_1, ..., \gamma_M), \; \gamma_j = \gamma_0, \; j = 1(1)\, M$$

and

$$\eta_0(t, \epsilon) = \epsilon M_3 e^{m(t-a)} + \epsilon M_4 [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] [e^{m(t-a)} - e^{-k(t-a)/\epsilon}] \qquad ...(4.18)$$

$$\gamma_0(t, \epsilon) = \epsilon M_5 e^{m(t-a)} + \epsilon M_6 [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] \, e^{m(t-a)} + [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] \, e^{-k(t-a)/\epsilon}. \qquad ...(4.19)$$

The vector function $\bar{z}^*$ so defined is nothing but a superfunction with respect to a solution $\bar{z}$ of the IVP (4.1). For,

$$
\begin{aligned}
P_i\bar{z}^* &= X_i'(t) + \eta_i'(t, \epsilon) - [u_i(t, \bar{X} + \bar{\eta}, \bar{Y} + \bar{\gamma}, \epsilon) - u_i(t, \bar{X}, \bar{Y}, \epsilon)] \\
&\qquad - u_i(t, \bar{X}, \bar{Y}, \epsilon) \\
&\geqslant -\epsilon M_7 + \eta_0' - l[\eta_0 + \gamma_0], \text{ by (4.13), (4.4) and (4.6)} \\
&= \epsilon e^{m(t-a)}[-M_7 + mM_3 - lM_3 - lM_5] \\
&\quad + \epsilon [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] e^{m(t-a)} [mM_4 - lM_4 - lM_6] \\
&\quad + [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] e^{-k(t-a)/\epsilon} [kM_4 + l\epsilon M_4 - l] \\
&\qquad \text{by (4.18) and (4.19).}
\end{aligned}
$$

Now choose M_4, M_5, M_6, and m such that

$$kM_4 - l > 0 \qquad \ldots(4.20)$$

$$kM_5 - M_8 - M_9 - lM_3 > 0 \qquad \ldots(4.21)$$

$$-lM_4 + kM_6 > 0 \qquad \ldots(4.22)$$

$$mM_3 - M_7 - lM_3 - lM_5 > 0 \text{ and } mM_4 - lM_4 - lM_6 > 0. \qquad \ldots(4.23)$$

With this choice, we have

$$P_i\bar{z}^* \geqslant 0 = P_i\bar{z}, \; i = 1(1)N.$$

Also we have,

$$
\begin{aligned}
P_{N+j}\bar{z}^* &= \epsilon Y_j'(t) + \epsilon\gamma_0'(t, \epsilon) - [v_j(t, \bar{X} + \bar{\eta}, \bar{Y} + \bar{\gamma}, \epsilon) - v_j(t, \bar{X}, \bar{Y} + \bar{\gamma}, \epsilon) \\
&\qquad + v_j(t, \bar{X}, \bar{Y} + \bar{\gamma}, \epsilon) - v_j(t, \bar{X}, \bar{Y}, \epsilon)] - v_j(t, \bar{X}, \bar{Y}, \epsilon) \\
&\geqslant -\epsilon M_9 - \epsilon M_8 + \epsilon\gamma_0' - l\eta_0 + k\gamma_0, \text{ by (4.14), (4.15), (4.5) - (4.7)} \\
&= \epsilon e^{m(t-a)}[-M_9 - M_8 + \epsilon mM_5 - lM_3 + kM_5] \\
&\quad + \epsilon [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] e^{m(t-a)} [\epsilon mM_6 - lM_4 + kM_5] \\
&\quad + [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |] e^{-k(t-a)/\epsilon} [-k + l\epsilon M_4 + k], \\
&\qquad \text{by (4.18) and (4.19).}
\end{aligned}
$$

Using (4.20) - (4.23), we have

$$P_{N+j}\bar{z}^* \geqslant 0 = P_{N+j}\bar{z}, \; j = 1(1)N.$$

Again,

$$x_i^*(a, \epsilon) = X_i(a) + \epsilon M_3 = A(0) + \epsilon M_3 \geqslant A_i(\epsilon) = x_i(a, \epsilon), \quad i = 1(1)\, N,$$

$$y_j^*(a, \epsilon) = Y_j(a) + \epsilon M_5 + \epsilon M_6 [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |]$$

$$+ [\max_{j=1(1)M} | B_j(\epsilon) - Y_j(a) |]$$

$$\geqslant Y_j(a) + B_j(\epsilon) - Y_j(a) = B_j(\epsilon) = y_j(a, \epsilon), \; j = 1(1)\, M.$$

That is, we have the following set of inequalities

$$\left. \begin{array}{l} P_i\bar{z} \leqslant P_i\bar{z}^*, \; R_i\bar{z} \leqslant R_i\bar{z}^*, \; i = 1(1)\, N \\ P_{N+j}\bar{z} \leqslant P_{N+j}\bar{z}^*, \; R_{N+j}\bar{z} \leqslant R_{N+j}\bar{z}^*, \; j = 1(1)\, M \end{array} \right\} \quad \ldots(4.24)$$

which, by Theorem 2.2, yield

$$\left. \begin{array}{l} x_i(t, \epsilon) \leqslant x_i^*(t, \epsilon), \quad i = 1(1)\, N, \quad t \in \bar{D} \\ y_j(t, \epsilon) \leqslant y_j^*(t, \epsilon), \quad j = 1(1)\, M, \quad t \in \bar{D}. \end{array} \right\} \quad \ldots(4.25)$$

Similarly one can show that

$$\left. \begin{array}{l} X_i(t) - \eta_0(t, \epsilon) \leqslant x_i(t, \epsilon), \; i = 1(1)\, N, \; t \in \bar{D} \\ Y_j(t) - \gamma_0(t, \epsilon) \leqslant y_j(t, \epsilon), \; j = 1(1)\, M, \; t \in \bar{D}. \end{array} \right\} \quad \ldots(4.26)$$

The inequalities (4.25) and (4.26) yield (4.10) and (4.11) which, in turn, lead to the following results :

$$\operatorname*{Lt}_{\epsilon \to 0+} x_i(t, \epsilon) = X_i(t), \; a \leqslant t \leqslant b, \; i = 1(1)\, N$$

$$\operatorname*{Lt}_{\epsilon \to 0+} y_j(t, \epsilon) = Y_j(t), \; a < t \leqslant b, \; j = 1(1)\, M.$$

5. Initial Value Problems for Higher Order Equations

The analysis of the previous section can be carried over to study the IVPs for higher order equations :

$$\left. \begin{array}{l} \epsilon y^{(n)} = f(t, y, y^{(1)}, \ldots, y^{(n-1)}, \epsilon), \; a < t \leqslant b \\ y(a, \epsilon) = A_1(\epsilon), \ldots, y^{(n-1)}(a, \epsilon) = A_n(\epsilon) \end{array} \right\} \quad \ldots(5.1)$$

where $\epsilon > 0$ is a small parameter and $y^{(n)} := d^n y/dt^n$,

$$A_i(\epsilon) = A_i(0) + O(\epsilon), \; i = 1(1)\, n.$$

The system associated with the IVP (5.1) is

$$\left.\begin{aligned} &P_1\bar{y} : = y_1' - y_2 = 0, \ldots, P_{n-1}\bar{y} : = y_{n-1}' - y_n = 0 \\ &P_n\bar{y} : = \epsilon y_n' - f(t, y_1, \ldots, y_n, \epsilon) = 0,\ t \in D \\ &R_i\bar{y} : = y_i(a, \epsilon) = A_i(\epsilon),\ i = 1(1)\,n \end{aligned}\right\} \qquad \ldots(5.2)$$

where $\quad \bar{y} : = (y_1, \ldots, y_n),\ y_1 : = y.$

It is obvious that results similar to that obtained for the IVP (4.1) can be derived for the IVP (5.1) through the IVP (5.2).

The degenerate problem for the IVP (5.1) is given by

$$\left.\begin{aligned} &u_1' - u_2 = 0, \ldots, u_{n-1}' - u_n = 0,\ f(t, u_1, \ldots, u_n, 0) = 0 \\ &u_i(a, 0) = A_i(0),\ i = 1(1)\,n - 1,\ u_1 : = u. \end{aligned}\right\} \qquad \ldots(5.3)$$

We shall make the following assumptions.

(I) $f(t, y_1, \ldots, y_n, \epsilon)$ is monotonically increasing in y_j,

$j = 1(1)\,n - 1$ with $y_j = y_{j-1}' = y^{(j-1)}$.

(II) There exists a function u satisfying (5.3) such that

$$f(t, u, \ldots, u^{(n-2)}, u^{(n-1)} + \gamma_0(t, \epsilon), \epsilon) - f(t, u, \ldots, u^{(n-1)}, \epsilon)$$
$$\leqslant -k\gamma_0(t, \epsilon),\ k > 0,\ \gamma_0(t, \epsilon) > 0,\ t \in D.$$

(III) For some constant $l > 0$

$$f(t, u + \eta_0, \ldots, u^{(n-2)} + \eta_0, u^{(n-1)} + \gamma_0, \epsilon)$$
$$- f(t, u, \ldots, u^{(n-2)}, u^{(n-1)} + \gamma_0, \epsilon) \leqslant l\eta_0,\ \eta_0 = \eta_0(t, \epsilon) > 0.$$

Theorem 5.1 — Consider the IVP (5.1) and let y and u be respectively solutions of the IVPs (5.1) and (5.3). Further assume that $f(t, u, \ldots, u^{(n-1)}, \epsilon) = O(\epsilon)$. Then under the assumptions I – III listed above, we have for $m > 0$,

$$\begin{aligned} |\ y^{(i)}(t, \epsilon) - u^{(i)}(t)\ | &\leqslant \epsilon M_3 e^{m(t-a)} \\ &+ \epsilon M_4\ |\ A_n(\epsilon) - u^{(n-1)}(a)\ |\ [e^{m(t-a)} \\ &- e^{-k(t-a)/\epsilon}],\ i = O(1)\,(n - 2) \\ &y^{(0)} : = y,\ u^{(0)} : = u. \end{aligned} \qquad \ldots(5.4)$$

$$
\begin{aligned}
| y^{(n-1)}(t, \epsilon) - u^{(n-1)}(t) | \leqslant\ & \epsilon M_5 e^{m(t-a)} \\
& + \epsilon M_6 \, | A_n(\epsilon) - u^{(n-1)}(a) | \, e^{m(t-a)} \\
& + | A_n(\epsilon) - u^{(n-1)}(a) | \, e^{-k(t-a)/\epsilon}, \ t \in D.
\end{aligned} \qquad \ldots(5.5)
$$

The proof of this theorem follows as in Theorem 4.1.

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