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A CHARACTERIZATION OF REGULAR POINTS FOR THE DIRICHLET PROBLEM IN A HARMONIC SPACE

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In a Brelot harmonic space with positive potentials, we obtain the following criterion for a point to be regular for the Dirichlet problem : "For a relatively compact open set X, a point $x \in \partial X$ is regular for the Dirichlet problem if and only if p_x^*, the potential function in the adjoint harmonic sheaf with the point support x is the limit of an increasing sequence of positive, bounded harmonic functions of the adjoint harmonic sheaf."

Let Ω be a harmonic space, that is a locally compact, non-compact, connected and locally connected space, provided with a sheaf of harmonic functions satisfying the axioms 1, 2, 3 of Brelot (1960) with the following additional conditions : (i) the constants are harmonic in Ω; (ii) Ω has a countable base; (iii) there exists a base of completely determined domains in Ω (Herve 1962, §11); (iv) there exists a potential > 0 in Ω and for every $x \in \Omega$, the potentials with support x in Ω are proportional; and (v) Ω satisfies axiom D. (Herve 1962, §25).

We note that the adjoint harmonic space associated with Ω also satisfies the axiom D.

In this paper, we obtain a characterization of the regular points for the Dirichlet problem in harmonic spaces, in terms of the quasi boundedness of potentials with point support in the adjoint harmonic space. An analogous result involving Green functions in R^n has been obtained by Kuran (1979).

We recall that, for a relatively compact open set $X \subset \Omega$, a point $x \in \partial X$ is said to be 'regular', if for every finite continuous function f on ∂X, the Dirichlet solution $H_f^X(y) \to f(x)$ as $y \to x$ in X.

We also recall that, a positive harmonic function h in an open set X is said to be 'quasi-bounded', if there exists an increasing sequence $\{h_n\}$ of positive, bounded harmonic functions in X, such that $\lim_{n \to \infty} h_n = h$ in X.

Let (w_0, x_0) be a fixed pair, where w_0 is a regular domain and $x_0 \in w_0$. For a given x, let p_x be the potential in Ω, with point support x, such that $\int p_x \, d\rho_{x_0}^{w_0} = 1$. Let $p_x^*(y) = p_y(x)$.

Lemma 1 — Let X be a relatively compact open set in Ω and u be a positive superharmonic function in Ω, which is quasi-bounded in X. Then $u = \bar{H}_u^X$ in X.

PROOF : Since u is superharmonic in Ω, $u \geqslant \bar{H}_u^X$ in X. There exists an increasing sequence $\{h_n\}$ of positive, bounded harmonic functions in X, such that $\lim\limits_{n\to\infty} h_n = u$ in X.

$h_n \leqslant u$ in X implies that $h_n \leqslant \bar{H}_u^X$, for all n, which yields $u \leqslant \bar{H}_u^X$ in X.

We conclude $u = \bar{H}_u^X$ in X.

For any two positive superharmonic functions u and v in an open set X, we denote by $u \wedge v$, the greatest harmonic minorant of $\inf(u, v)$ in X.

Proposition 2 — Let X be an open set in Ω and h be a positive harmonic function in X. Then h is quasi-bounded in X if and only if $h = \lim\limits_{n\to\infty} (h \wedge n)$.

PROOF : It is obvious that if $h = \lim\limits_{n\to\infty} (h \wedge n)$, then h is quasi-bounded in X.

Conversely, assume that h is quasi-bounded in X. Then, there exists an increasing sequence h_n of positive, bounded harmonic functions in X such that $\lim\limits_{n\to\infty} h_n = h$. For each n, there exists an integer $p_0 > 0$ such that $h_n \leqslant h \wedge p$ for all $p \geqslant p_0$, which gives, $h_n \leqslant \lim\limits_{n\to\infty} (h \wedge n)$. Hence $h \leqslant \lim\limits_{n\to\infty} (h \wedge n)$. We conclude $h = \lim\limits_{n\to\infty} (h \wedge n)$.

Proposition 3 — Let X be an open set and $\{D_\alpha\}$ be the family of connected components of X. Then a positive harmonic function h is quasi-bounded in X if and only if it is quasi-bounded in each of the D_α.

PROOF : It is obvious that when h is quasi-bounded in X, it is quasi-bounded on each D_α.

Conversely, assume that h is quasi-bounded on each D_α. Then $h = \lim\limits_{n\to\infty} (h \wedge n)$ in each D_α. As $(h \wedge n)$ on $D_\alpha = (h \wedge n)$ on X, on each D_α, we conclude $h = \lim\limits_{n\to\infty} (h \wedge n)$ on X. h is therefore quasi-bounded in X.

Theorem 4 — Let X be a relatively compact open set in Ω and $x \in \partial X$. Then x is regular for X if and only if p_x^* is quasi-bounded in X.

PROOF : First assume that p_x^* is quasi-bounded in X. Let D be a connected component of X with $x \in \partial D$. Then p_x^* is quasi-bounded in D.

By Lemma 1, $p_x^* = \bar{H}^{*X}_{(p_x^*)} = \hat{R}^{*\Omega-X}_{p_x^*}$ in X. As $R^{*\Omega-X}_{p_x^*} = p_x^*$ in $\Omega - X$, we get $p_x^* = R^{*\Omega-X}_{p_x^*} = \hat{R}^{*\Omega-X}_{p_x^*}$ in Ω.

We now prove that $\lim\limits_{X \ni y \to x} \bar{H}^X_{p_z}(y) = p_z(x)$ for all $z \in X$. For this note,

$$\liminf_{X \ni y \to x} \bar{H}^X_{p_z}(y) = \liminf_{X \ni y \to x} \hat{R}^{\Omega-X}_{p_z}(y) \geqslant \hat{R}^{\Omega-X}_{p_z}(x)$$

$$\geqslant \hat{R}^{*\Omega-X}_{p_x^*}(z) \geqslant p_x^*(z) \geqslant p_z(x).$$

Also, $$\limsup_{X \ni y \to x} \bar{H}^X_{p_z}(y) \leqslant \limsup_{X \ni y \to x} p_z(y) \leqslant p_z(x).$$

Hence $$\lim_{X \ni y \to x} \bar{H}^X_{p_z}(y) = p_z(x) \text{ for all } z \in X.$$

Fix now $z \in D$. The function $p_z - \bar{H}^X_{p_z} > 0$ is super-harmonic in D and

$$\lim_{X \ni y \to x} (p_z - \bar{H}^X_{p_z})(y) = 0.$$

We conclude by applying Theorem 22 of Brelot (1960) that x is regular in D and hence in X.

Conversely, assume that x is regular for X. Then for any superharmonic function $v \geqslant 0$ in Ω, $\hat{R}^{\Omega-X}_v(x) = v(x)$. [Corollary to Theorem 19 of Brelot (1960)]. In particular, $\hat{R}^{\Omega-X}_{p_z}(x) = p_z(x)$ for all $z \in X$, which gives $\hat{R}^{*\Omega-X}_{p_x^*}(z) = p_x^*(z)$. Define $h_n^* = \inf(p_x^*, n)$. $h_n^* \geqslant 0$ is superharmonic and bounded in the adjoint space. Also $\{h_n^*\}$ increases. Hence $\left\{\bar{H}^{*X}_{h_n^*}\right\}$ is an increasing sequence of bounded harmonic functions such that $\lim\limits_{n \to \infty} \bar{H}^{*X}_{h_n^*} = p_x^*$ in X, in the adjoint space. We conclude p_x^* is quasi-bounded in X.

Remark : In the special case of self adjoint harmonic spaces, the above theorem reduces to, x is regular if and only if p_x is quasi-bounded in X.

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