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COEFFICIENT ESTIMATES FOR SOME CLASSES OF SPIRAL-LIKE FUNCTIONS*

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The authors introduce the class $F_\lambda(\alpha, \beta)$ of regular functions $f(z)=z+a_2z^2+\ldots$, defined in the unit disk $E = \{z : | z | < 1\}$ and satisfying the condition

$$\left|\frac{H(f(z)) - 1}{H(f(z)) + 1}\right| < \lambda$$

where

$$H(f(z)) = \frac{e^{i\alpha} \dfrac{zf'(z)}{f(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}$$

for all z in E and the class $H_\lambda(\alpha, \beta)$ of regular functions $g(z) = (1/z) + a_1 z + \ldots$, defined in the punctured disk $E' = \{z : 0 < | z | < 1\}$ and satisfying the condition

$$\left|\frac{H_1(g(z)) - 1}{H_1(g(z)) + 1}\right| < \lambda$$

where

$$H_1(g(z)) = \frac{- e^{i\alpha} \dfrac{zg'(z)}{g(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha},$$

for all z in E', where $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [0, 1)$ and $\lambda \in (0, 1]$. Using the technique of Clunie (1959) the authors obtain the sharp coefficient estimates for the class $F_\lambda(\alpha, \beta)$ and $H_\lambda(\alpha, \beta)$.

1. INTRODUCTION

Let A denote the class of functions $f(z)$ which are analytic in the unit disk $E = \{z : | z | < 1\}$ and satisfy the conditions $f(0) = 0$ and $f'(0) = 1$. For

$$\alpha \in (-\pi/2, \pi/2)$$

and $\beta \in [0, 1)$, let $F(\alpha, \beta)$ denote the class of functions $f(z) \in A$ which satisfy the condition

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$$\text{Re } e^{i\alpha} \frac{zf'(z)}{f(z)} > \beta \cos \alpha \qquad \ldots(1.1)$$

for all z in E. We call functions of $F(\alpha, \beta)$ as α-spiral functions of order β. Clearly $F(\alpha, \beta) \subset F(\alpha)$ and $F(\alpha, 0) \equiv F(\alpha)$, where $F(\alpha)$ is the well-known class of α-spiral functions introduced by Špaček (1933). The class $F(\alpha, \beta)$ was introduced and studied by Libera (1967).

For $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [0, 1)$ and $\lambda \in (0, 1]$, let $F_\lambda(\alpha, \beta)$ denote the class of functions $f(z)$ belonging to A which satisfy the condition

$$\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda \qquad \ldots(1.2)$$

for $z \in E$, where

$$H(f(z)) = \frac{e^{i\alpha} \dfrac{zf'(z)}{f(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}. \qquad \ldots(1.3)$$

It is easy to see that $F_\lambda(\alpha, \beta) \subset F(\alpha, \beta)$ for every $\lambda \in (0, 1]$ and $F_1(\alpha, \beta) \equiv F(\alpha, \beta)$.

In this paper we use a method of Clunie (1959) to obtain sharp bounds for the coefficients of functions of $F_\lambda(\alpha, \beta)$. In the rest of the paper we always assume that $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [0, 1)$ and $\lambda \in (0, 1]$.

2.

The following lemma gives a representation formula for functions of $F_\lambda(\alpha, \beta)$.

Lemma 1 — If $f(z) \in A$, then $f(z) \in F_\lambda(\alpha, \beta)$ if and only if

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \, \frac{1 + (2\beta - 1)\, w(z)}{1 + w(z)} + i \sin \alpha \qquad \ldots(2.1)$$

for $z \in E$ for some $w(z)$ analytic in E and satisfying $w(0) = 0$ and $| w(z) | < \lambda$ for $z \in E$.

PROOF : If $f(z)$ is given by (2.1), then

$$H(f(z)) = \frac{1 - w(z)}{1 + w(z)} \text{ so that } \frac{H(f(z)) - 1}{H(f(z)) + 1} = - w(z)$$

and so (1.2) holds. Thus $f(z) \in F_\lambda(\alpha, \beta)$.

If $f(z) \in F_\lambda(\alpha, \beta)$, then (1.2) holds.

Defining $w(z) = \dfrac{1 - H(f(z))}{1 + H(f(z))}$ we obtain (2.1).

Lemma 2 — If m is a natural number $\geqslant 3$, then

$$\frac{\cos^2 \alpha}{(m-1)^2}\left\{4\lambda^2(1-\beta)^2 + \sum_{k=2}^{m-1}\left[(k+1-2\beta)^2\lambda^2 + (k-1)^2\lambda^2\tan^2\alpha - (k-1)^2\sec^2\alpha\right]\prod_{j=0}^{k-2} u_j\right\} = \prod_{j=0}^{m-2} u_j, \qquad \ldots(2.2)$$

where

$$u_j = \frac{\lambda^2 \mid 2(1-\beta)\cos\alpha e^{-i\alpha} + j \mid^2}{(j+1)^2} \quad \text{for } j = 0, 1, 2, \ldots . \qquad \ldots(2.3)$$

PROOF: We prove the lemma by induction on m. For $m = 3$, (2.2) is easily verified directly. Suppose, now, that (2.2) holds for $m = p - 1$ for some $p \geqslant 4$. Then, for $m = p$, the left member of (2.2) reduces to

$$\frac{\cos^2 \alpha}{(p-1)^2}\left\{4\lambda^2(1-\beta)^2 + \sum_{k=2}^{p-2}\left[(k+1-2\beta)^2\lambda^2 + (k-1)^2\lambda^2\tan^2\alpha - (k-1)^2\sec^2\alpha\right]\prod_{j=0}^{k-2} u_j + \left[(p-2\beta)^2\lambda^2 + (p-2)^2\lambda^2\tan^2\alpha - (p-2)^2\sec^2\alpha\right]\prod_{j=0}^{p-3} u_j\right\}$$

$$= \frac{1}{(p-1)^2}\left\{(p-2)^2\prod_{j=0}^{p-3} u_j + \left[(p-2\beta)^2\lambda^2\cos^2\alpha + (p-2)^2 \times \lambda^2\sin^2\alpha - (p-2)^2\right]\prod_{j=0}^{p-3} u_j\right\} \text{ (by the inductive hypothesis).}$$

$$= \frac{\lambda^2}{(p-1)^2}\left\{(p-2\beta)^2\cos^2\alpha + (p-2)^2\sin^2\alpha\right\}\prod_{j=0}^{p-3} u_j$$

$$= \prod_{j=0}^{p-2} u_j.$$

Thus (2.2) holds for $m = p$ which proves Lemma 2.

Theorem 1 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F_\lambda(\alpha, \beta)$, then

$$| a_n | \leqslant \prod_{j=0}^{n-2} u_j^{1/2} \qquad \ldots(2.4)$$

for $n = 2, 3, \ldots$, where u_j is defined by (2.3) for $j = 0, 1, 2, \ldots$. The result is sharp.

PROOF : We have by Lemma 1,

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \, \frac{1 + (2\beta - 1)\,\omega(z)}{1 + \omega(z)} + i \sin \alpha$$

for $z \in E$ where $\omega(z)$ is analytic in E, $\omega(0) = 0$ and $| \omega(z) | < \lambda$ for $z \in E$. This yields

$$\{e^{i\alpha} \sec \alpha \,.\, zf'(z) + (1 - 2\beta - i \tan \alpha)\, f(z)\}\, \omega(z)$$

$$= (1 + i \tan \alpha)\, f(z) - e^{i\alpha} \sec \alpha \,.\, zf'(z).$$

That is,

$$\left\{ \sum_{k=1}^{\infty} [ke^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha]\, a_k \,.\, z^k \right\} \omega(z)$$

$$= - (1 + i \tan \alpha) \sum_{k=2}^{\infty} (k - 1)\, a_k z^k, \text{ where } a_1 = 1.$$

Since $\omega(z)$ is of the form $\omega(z) = \sum_{k=1}^{\infty} b_k z^k$ we obtain for $n \geqslant 2$,

$$\left\{ \sum_{k=1}^{n-1} [ke^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha)]\, a_k z^k \right\} \omega(z)$$

$$= - (1 + i \tan \alpha) \sum_{k=2}^{n} (k - 1)\, a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k \qquad \ldots(2.5)$$

where $\sum_{k=n+1}^{\infty} d_k z^k$ converges in E.

Denoting the right member of (2.5) by $G(z)$ and the factor multiplying $\omega(z)$ in the left member of (2.5) by $F(z)$, (2.5) assumes the form

$$G(z) = F(z)\, \omega(z) \text{ for } z \in E.$$

Since $| \omega(z) | < \lambda$ for $z \in E$ this yields for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} | G(re^{i\theta}) |^2 \, d\theta \leqslant \lambda^2 \frac{1}{2\pi} \int_0^{2\pi} | F(re^{i\theta}) |^2 \, d\theta,$$

whence, using the definitions of $G(z)$ and $F(z)$

$$\sec^2 \alpha \sum_{k=2}^{n} (k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k}$$

$$\leqslant \lambda^2 \left\{ \sum_{k=1}^{n-1} |ke^{i\alpha} \sec \alpha + (1 - 2\beta - i \tan \alpha)|^2 |a_k|^2 r^{2k} \right\}.$$

Letting $r \to 1$, we obtain for $n \geqslant 2$

$$\sec^2 \alpha \sum_{k=2}^{n} (k-1)^2 |a_k|^2$$

$$\leqslant \lambda^2 \left\{ \sum_{k=1}^{n-1} [(k+1-2\beta)^2 + (k-1)^2 \tan^2 \alpha] |a_k|^2 \right\}$$

which may be written as

$$\sec^2 \alpha (n-1)^2 |a_n|^2 \leqslant \sum_{k=1}^{n-1} [(k+1-2\beta)^2 \lambda^2 + (k-1)^2 \lambda^2 \tan^2 \alpha - (k-1)^2 \sec^2 \alpha] |a_k|^2$$

$$|a_n|^2 \leqslant \frac{\cos^2 \alpha}{(n-1)^2} \sum_{k=1}^{n-1} [(k+1-2\beta)^2 \lambda^2 + (k-1)^2 \lambda^2 \tan^2 \alpha - (k-1)^2 \sec^2 \alpha] |a_k|^2 \text{ for } n = 2, 3, \ldots . \quad \ldots(2.6)$$

For $n = 2$, (2.6) yields $|a_2|^2 \leqslant 4(1-\beta)^2 \lambda^2 \cos^2 \alpha = u_0$ which proves (2.4) for $n = 2$.

We now prove (2.4) for all $n \geqslant 2$ by induction on n. Suppose (2.4) holds for $n \leqslant p - 1$ for some $p \geqslant 3$. Then for $n = p$, (2.6) yields,

$$|a_p|^2 \leqslant \frac{\cos^2 \alpha}{(p-1)^2} \sum_{k=1}^{p-1} [(k+1-2\beta)^2 \lambda^2 + (k-1)^2 \lambda^2 \tan^2 \alpha - (k-1)^2 \sec^2 \alpha] |a_k|^2$$

$$\leqslant \frac{\cos^2 \alpha}{(p-1)^2} \left\{ 4(1-\beta)^2 \lambda^2 + \sum_{k=2}^{p-1} \left[(k+1-2\beta)^2 \lambda^2 + (k-1)^2 \lambda^2 \tan^2 \alpha - (k-1)^2 \sec^2 \alpha \right] \prod_{j=2}^{k-2} u_j \right\}$$

$$= \prod_{j=0}^{p-2} u_j, \text{ by Lemma 2.}$$

So (2.4) holds for all $n = p$. Thus (2.4) holds for all $n \geqslant 2$.

Equality holds in (2.4) for each $n \geqslant 2$ for the function $f(z)$ in A defined by (2.1) with $\omega(z) = \lambda z$.

This completes the proof of Theorem 1.

For $\lambda = 1$, Theorem 1 reduces to a result of Libera (1967). For $\lambda = 1$ and $\alpha = 0$ in Theorem 1, we obtain a result of Schild (1965). For $\lambda = 1$ and $\beta = 0$, Theorem 1 yields a result of Zamorski (1962).

3. Meromorphic Spiral-like Functions

Let E' denote the punctured disk $\{z : 0 < |z| < 1\}$. For $\alpha \in (-\pi/2, \pi/2)$ and $\beta \in [0, 1)$, let $H(\alpha, \beta)$ denote the family of functions $g(z)$ analytic in E' and of the form

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad \text{...(3.1)}$$

for $z \in E'$ which satisfy the condition

$$\text{Re}\left\{-e^{i\alpha} z \frac{g'(z)}{g(z)}\right\} > \beta \cos \alpha. \quad \text{...(3.2)}$$

Functions of the class $H(\alpha, \beta)$ are called meromorphic α-spiral-like of order β. Kaczmarski (1969) obtained sharp coefficient estimates for such functions.

Now, for $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [0, 1)$ and $\lambda \in (0, 1]$, let $H_\lambda(\alpha, \beta)$ denote the class of all functions $g(z)$ analytic in E' and of the form (3.1) which satisfy the condition

$$\left|\frac{H_1(g(z)) - 1}{H_1(g(z)) + 1}\right| < \lambda \quad \text{...(3.3)}$$

for $z \in E'$, where

$$H_1(g(z)) = \frac{-e^{i\alpha} z \dfrac{g'(z)}{g(z)} - \beta \cos \alpha - i \sin \alpha}{(1 - \beta) \cos \alpha}.$$

It is clear that $H_\lambda(\alpha, \beta) \subset H(\alpha, \beta)$ for all $\lambda \in (0, 1]$ and $H_1(\alpha, \beta) \equiv H(\alpha, \beta)$.

It can be shown that $g(z) \in H_\lambda(\alpha, \beta)$ if and only if

$$-e^{i\alpha} z \frac{g'(z)}{g(z)} = \cos \alpha \frac{1 + (2\beta - 1)\,\omega(z)}{1 + \omega(z)} + i \sin \alpha$$

for $z \in E'$, for some function $\omega(z)$ analytic in E and satisfying $\omega(0) = 0$ and $|\omega(z)| < \lambda$ for $z \in E$.

Proceeding as in the proof of Theorem 1, we can prove the following

Theorem 2 — If

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in H_\lambda(\alpha, \beta), \text{ then}$$

$$|b_n| \leqslant \frac{2\lambda(1-\beta)\cos\alpha}{n+1} \quad \text{for } n = 1, 2, 3, \ldots . \qquad \ldots(3.4)$$

The result is sharp.

In fact equality holds in (3.4) for fixed n for the function

$$g(z) = \frac{1}{z}(1 - \lambda z^{n+1})^{2(1-\beta)\cos\alpha e^{-i\alpha}/(n+1)}$$

$$= \frac{1}{z} - \frac{2\lambda(1-\beta)\cos\alpha e^{-i\alpha}}{n+1} z^n + \ldots .$$

For $\lambda = 1$, Theorem 2 yields a result of Kaczmarski (1969).

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