Indian J. pure appl. Math., **11**(8) : 1021–1038, August 1980

SOME SUMMATION FORMULAE OF BASIC HYPERGEOMETRIC SERIES

A. VERMA AND V. K. JAIN

Department of Mathematics, University of Roorkee, Roorkee 247672

(*Received* 3 *December* 1979)

Recently Verma and Jain (1980) had obtained some transformations between the basic hypergeometric series on different bases. Using these transformations a number of interesting summation formulae for basic and ordinary hypergeometric series are obtained.

§1. Recently the authors obtained some transformations between basic hypergeometric series on different bases and had shown how they could be used for obtaining identities of Rogers-Ramanujan type related to the moduli 11, 13, 17, 19, 22, 26, 33, 34, 38, etc. (see Verma and Jain 1980). In this paper we would examine the following four of those transformations:

$$ {}_{10}\phi_9\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ x,\ -x,\ y,\ -y,\ -q^{-n},\ q^{-n};\ q;\ \dfrac{-a^3q^{3+2n}}{bx^2y^2} \\ \sqrt{a},\ -\sqrt{a},\ \dfrac{aq}{b},\ \dfrac{aq}{x},\ \dfrac{-aq}{x},\ \dfrac{aq}{y},\ \dfrac{-aq}{y},\ -aq^{1+n},\ aq^{1+n} \end{matrix}\right] $$

$$ = \frac{[a^2q^2;\ q^2]_n \left[\dfrac{a^2q^2}{x^2y^2};\ q^2\right]_n}{\left[\dfrac{a^2q^2}{x^2};\ q^2\right]_n \left[\dfrac{a^2q^2}{y^2};\ q^2\right]_n}\ {}_5\phi_4\left[\begin{matrix} x^2,\ y^2,\ \dfrac{-aq}{b},\ \dfrac{-aq^2}{b},\ q^{-2n};\ q^2;\ q^2 \\ -aq,\ -aq^2,\ \dfrac{a^2q^2}{b^2},\ \dfrac{x^2y^2q^{-2n}}{a^2} \end{matrix}\right], \quad \ldots(1.1) $$

$$ {}_{10}\phi_9\left[\begin{matrix} a,\ q^2\sqrt{a},\ -q^2\sqrt{a},\ b,\ x,\ xq,\ y,\ yq,\ q^{-n+1},\ q^{-n};\ q^2;\ \dfrac{a^3q^{3+2n}}{bx^2y^2} \\ \sqrt{a},\ -\sqrt{a},\ \dfrac{aq^2}{b},\ \dfrac{aq^2}{x},\ \dfrac{aq}{x},\ \dfrac{aq^2}{y},\ \dfrac{aq}{y},\ aq^{1+n},\ aq^{2+n} \end{matrix}\right] $$

$$ = \frac{[aq;\ q]_n \left[\dfrac{aq}{xy};\ q\right]_n}{\left[\dfrac{aq}{x};\ q\right]_n \left[\dfrac{aq}{y};\ q\right]_n}\ {}_5\phi_4\left[\begin{matrix} x,\ y,\ \sqrt{\dfrac{aq}{b}},\ -\sqrt{\dfrac{aq}{b}},\ q^{-n};\ q;\ q \\ \sqrt{aq},\ -\sqrt{aq},\ \dfrac{aq}{b},\ \dfrac{xyq^{-n}}{a} \end{matrix}\right] \quad \ldots(1.2) $$

$$ {}_{12}\phi_{11}\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ x,\ wx,\ w^2x,\ y,\ wy,\ w^2y,\ q^{-n},\ wq^{-n},\ w^2q^{-n};\ q;\ a^4q^{4+3n}/x^3y^3 \\ \sqrt{a},\ -\sqrt{a},\ \dfrac{aq}{x},\ \dfrac{aqw^2}{x},\ \dfrac{aqw}{x},\ \dfrac{aq}{y},\ \dfrac{aqw^2}{y},\ \dfrac{aqw}{y},\ aq^{1+n},\ w^2aq^{1+n},\ waq^{1+n} \end{matrix}\right] = $$

(*equation continued on p.* 1022)

$$= \frac{[a^3q^3;\, q^3]_n\, [a^3q^3/x^3y^3;\, q^3]_n}{[a^3q^3/x^3;\, q^3]_n\, [a^3q^3/y^3;\, q^3]_n}$$

$$\times\, {}_6\phi_5 \left[\begin{matrix} aq,\, aq^2\, aq^3,\, x^3,\, y^3,\, q^{-3n};\, q^3;\, q^3 \\ (aq)^{3/2},\, -(aq)^{3/2},\, a^{3/2}q^3,\, -a^{3/2}q^3,\, x^3y^3q^{-3n}/a^3 \end{matrix} \right] \quad ...(1.3)$$

$${}_{12}\phi_{11} \left[\begin{matrix} a,\, q^3\sqrt{a},\, -q^3\sqrt{a},\, x,\, xq,\, xq^2,\, y,\, yq,\, yq^2,\, q^{-n},\, q^{-n+1},\, q^{-n+2};\, q^3;\, a^4q^{3+3n}/x^3y^3 \\ \sqrt{a},\, -\sqrt{a},\, \frac{aq^3}{x},\, \frac{aq^2}{x},\, \frac{aq}{x},\, \frac{aq^3}{y},\, \frac{aq^2}{y},\, \frac{aq}{y},\, aq^{3+n},\, aq^{2+n},\, aq^{1+n} \end{matrix} \right]$$

$$= \frac{[aq;\, q]_n \left[\frac{aq}{xy};\, q\right]_n}{\left[\frac{aq}{x};\, q\right]_n \left[\frac{aq}{y};\, q\right]_n} \cdot {}_6\phi_5 \left[\begin{matrix} a^{1/3},\, wa^{1/3},\, w^2a^{1/3},\, x,\, y,\, q^{-n};\, q;\, q \\ \sqrt{a},\, -\sqrt{a},\, \sqrt{aq},\, -\sqrt{aq},\, xyq^{-n}/a \end{matrix} \right] \quad ...(1.4)$$

[for definitions and notations see Verma and Jain (1980)] and deduce a number of known as well as new summation formulae for basic hypergeometric series. In §2 we obtain from (1.1) a q-analogue of terminating version of Dixon's theorem different from the known ones (Jackson 1941, Bailey 1950, Carlitz 1969), q-analogue of terminating version of Watson's theorem (Slater 1966, Bailey 1953) together with a number of summation theorems for terminating Saalschützian ${}_4\phi_3$ which differ from the terminating version of summation theorems of Dixon and Watson. In §3 we deduce from (1.2) two summation formulae one for terminating Saalschützian ${}_4\phi_3$ and the other for terminating Saalschützian ${}_5\phi_4$.

In §4 we use (1.3) and (1.4) to show that they not only yield the two summation formulae of Andrews (1979) but also give three different summation theorems for terminating Saalschützian ${}_5\phi_4$. One of these results yields, on proceeding to the limit, the following interesting summation formula [c.f. Andrews 1979, (1.12)]

$${}_3F_2 \left[\begin{matrix} x,\, 3x+4+n,\, -n;\, \frac{3}{4} \\ \frac{3}{2}(x+1),\, \frac{3}{2}x+2 \end{matrix} \right] = \frac{(1)_n\, (2x+4)_n\, (x+2)_m\, (x+3)_{3m}}{(1+x)_n\, (3x+4)_n\, (1)_m\, (2x+4)_{3m}} \quad ...(1.5)$$

$\left(\text{where } m \text{ is the greatest integer} \leqslant \frac{n}{3}\right)$.

§2. In (1.1) setting $b = -aq^{1+n}$ and then transforming the resulting terminating well-poised ${}_8\phi_7$ on the left-hand side by the Watson's q-analogue of Whipple's formula (Bailey 1935):

$${}_8\phi_7 \left[\begin{matrix} a,\, q\sqrt{a},\, -q\sqrt{a},\, b,\, c,\, d,\, e,\, q^{-n};\, q;\, \frac{a^2q^{2+2n}}{bcde} \\ \sqrt{a},\, -\sqrt{a},\, \frac{aq}{b},\, \frac{aq}{c},\, \frac{aq}{d},\, \frac{aq}{e},\, aq^{1+n} \end{matrix} \right] =$$

(equation continued on p. 1023)

$$= \frac{[aq;q]_n \left[\frac{aq}{de};q\right]_n}{\left[\frac{aq}{d};q\right]_n \left[\frac{aq}{e};q\right]_n} {}_4\phi_3 \left[\begin{matrix} \frac{aq}{bc}, d, e, q^{-n}; q; q \\ \frac{deq^{-n}}{a}, \frac{aq}{b}, \frac{aq}{c} \end{matrix}\right] \qquad \ldots(2.1)$$

(with $b = -x$, $c = -y$, $d = x$, $e = y$) we get a transformation between a Saalschützian ${}_4\phi_3$ on base q and a Saalschützian ${}_4\phi_3$ on the base q^2, viz.,

$${}_4\phi_3 \left[\begin{matrix} \frac{aq}{xy}, x, y, q^{-n}; q; q \\ \frac{xyq^{-n}}{a}, -\frac{aq}{x}, -\frac{aq}{y} \end{matrix}\right]$$

$$= \frac{[-aq;q]_n [-aq/xy;q]_n}{[-aq/x;q]_n [-aq/y;q]_n} {}_4\phi_3 \left[\begin{matrix} x^2, y^2, q^{-n+1}, q^{-n}; q^2; q^2 \\ -aq, -aq^2, \frac{x^2y^2q^{-2n}}{a^2} \end{matrix}\right]. \qquad \ldots(2.2)$$

In (2.2) replacing a and x by $-a$ and $-x$ respectively and proceeding to the limit in the usual way (i.e. replacing a, x, y by q^a, q^x, q^y respectively and then letting $q \to 1$), we obtain a transformation between nearly-poised ${}_3F_2$ and a terminating Saalschützian ${}_4F_3$, which is a special case of a known result of Bailey [1935, 4.7(1)] (obtained on setting $w = c = -n - a + y, b = 1 + a - x - y, d = y$).

Next, (2.2) for $a = -q^{-n}$ reduces to the following q-analogue of terminating version of Dixon's theorem (Slater 1966, 2.3.3.5) different from the two known q-analogues of Dixon's theorem due to Jackson (1941) (see also Bailey 1950) and Carlitz (1969):

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, -\frac{q^{1-n}}{xy}, x, y; q; q \\ -xy, \frac{q^{1-n}}{x}, \frac{q^{1-n}}{y} \end{matrix}\right]$$

$$= \begin{cases} 0 \text{ if } n = 2m+1 \\ \dfrac{[x^2;q^2]_m [y^2;q^2]_m [xy;q]_{2m} [q;q^2]_m}{[x^2y^2;q^2]_m [x;q]_{2m} [y;q]_{2m}} & \text{if } n = 2m. \end{cases} \qquad \ldots(2.3)$$

Indeed (2.3) is equivalent to the known q-analogue of Watson's theorem due to Andrews (1976) to which it reduces on transforming the ${}_4\phi_3$ on the left-hand side by using the formula (Sears 1951)

$${}_4\phi_3 \left[\begin{matrix} a, b, c, q^{-n}; q; q \\ e, g, h \end{matrix}\right] = \frac{[g/c;q]_n [eg/ab;q]_n}{[g;q]_n [eg/abc;q]_n}$$

$$\times {}_4\phi_3 \left[\begin{matrix} \frac{e}{a}, \frac{e}{b}, c, q^{-n}; q; q \\ e, cq^{1-n}/g, cq^{1-n}/h \end{matrix}\right] \qquad \ldots(2.4)$$

(provided $abcq^{1-n} = egh$) with $a = x$, $b = -q^{1-n}/xy$, $c = y$, $e = q^{1-n}/x$, $g = -xy$, $h = q^{1-n}/y$.

On the other hand (2.2) for $a = -q^{-n-1}$, yields a summation formula for the terminating half-poised Saalschützian ${}_4\phi_3$

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{-n}/xy, x, y; q; q \\ -xyq, q^{-n}/x, q^{-n}/y \end{matrix}\right] \\
&= \frac{[q;q]_n\,[xyq;q]_n}{[xq;q]_n\,[yq;q]_n}\,{}_2\phi_1\left[\begin{matrix} x^2, y^2; q^2; q^2 \\ x^2y^2q^2 \end{matrix}\right] \text{to } (m+1) \text{ terms} \\
&\qquad \text{(where } m \text{ is the greatest integer} \leqslant n/2) \\
&= \frac{[q;q]_n\,[xyq;q]_n\,[x^2q^2;q^2]_m\,[y^2q^2;q^2]_m}{[xq;q]_n\,[yq;q]_n\,[x^2y^2q^2;q^2]_m\,[q^2;q^2]_m}. \qquad \text{...(2.5)}
\end{aligned}
$$

The last line is written using a summation theorem due to Agarwal [1953, 3(iii)] (Here it may be worthwhile to note that the two summation formulas for truncated hypergeometric series due to Anderson (1953) are special cases of a known result of Bailey (1932) whereas the q-analogue of these formulas due to Gould (1961) are special cases of the aforesaid result of Agarwal [1953, 3(ii)]).

(2.5) on proceeding to the limit yields :

$$
{}_3F_2\left[\begin{matrix} -n, x, y; \\ -n-x, -n-y \end{matrix}\right] = \frac{(1)_n\,(1+x+y)_n\,(1+x)_m\,(1+y)_m}{(1+x)_n\,(1+y)_n\,(1)_m\,(1+x+y)_m} \qquad \text{...(2.6)}
$$

while, in (2.5) replacing x by $-x/y$ and then proceeding to the limit, we get another summation theorem

$$
{}_3F_2\left[\begin{matrix} -n, -n-x, y; \\ 1+x, -n-y \end{matrix}\right] = \frac{(1)_n\,(1+x-y)_m\,(1+y)_m}{(1+y)_n\,(1+x)_m\,(1)_m}. \qquad \text{...(2.7)}
$$

The first being a summation formula for terminating nearly-poised ${}_3F_2$ of the first kind whereas the other is a summation formula for nearly-poised ${}_3F_2$ of the second kind.

In (2.5) letting $y \to 0$, we have a summation formula for a terminating ${}_2\phi_1$, very similar to the q-analogue of Kummer's summation theorem for ${}_2F_1[-1]$ [Andrews 1973, 1.7], viz.,

$$
{}_2\phi_1\left[\begin{matrix} q^{-n}, x; q; -q/x \\ q^{-n}/x \end{matrix}\right] = \frac{[q;q]_n\,[x^2q^2;q^2]_m}{[xq;q]_n\,[q^2;q^2]_m} \qquad \text{...(2.8)}
$$

on the other hand (2.5) for $y \to \infty$, yields

$$
{}_2\phi_1\left[\begin{matrix} q^{-n}, x; q; -1/x \\ q^{-n}/x \end{matrix}\right] = \frac{[q;q]_n\,[x^2q^2;q^2]_m\,x^{n-2m}}{[xq;q]_n\,[q^2;q^2]_m}. \qquad \text{...(2.9)}
$$

Now, transforming the ${}_4\phi_3$ on the left-hand side of (2.5) by (2.4) with $a = x$, $b = y$, $c = -q^{-n}/xy$, $e = -xyq$, $g = q^{-n}/x$, $h = q^{-n}/y$, we get the following summation formula (which may be also obtained from (2.5) by writing the series in the reverse order) for terminating half-poised Saalschützian ${}_4\phi_3$:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, -q^{-n}/xy, xq, yq; q; q \\ -xyq, q^{1-n}/x, q^{1-n}/y \end{matrix}\right]
$$

$$
= \frac{(-)^n [q; q]_n [xyq; q]_n [x^2q^2; q^2]_m}{q^n [x; q]_n [y; q]_n [x^2y^2q^2; q^2]_m} \frac{[y^2q^2; q^2]_m}{[q^2; q^2]_m}. \qquad \text{...(2.10)}
$$

(2.10) as a limiting case yields the following formula (which could also be obtained from (2.6) by writing the series in the reverse order)

$$
{}_3F_2\left[\begin{matrix} -n, 1 + x, 1 + y; \\ 1 - n - x, 1 - n - y \end{matrix}\right]
$$

$$
= \frac{(-)^n (1)_n (1 + x + y)_n (1 + x)_m (1 + y)_m}{(x)_n (y)_n (1 + x + y)_m (1)_m}. \qquad \text{...(2.11)}
$$

Similarly writing the series in (2.7), (2.8) and (2.9) in the reverse order we could obtain summation formulae for

$$
{}_3F_2\left[\begin{matrix} -n, -n - x, 1 + y; \\ 1 + x, 1 - n - y \end{matrix}\right], \quad {}_2\phi_1\left[\begin{matrix} q^{-n}, xq; q; -\dfrac{1}{x} \\ q^{1-n}/x \end{matrix}\right]
$$

and
$$
{}_2\phi_1\left[\begin{matrix} q^{-n}, xq; q; -q/x \\ q^{1-n}/x \end{matrix}\right]
$$

respectively. It may be pointed out that the first of these summation formulae could have been also obtained from (2.10) by first replacing x by $-x/y$ and then proceeding to the limits, whereas the other two could have also been obtained from (2.10) by letting $y \to 0$ and $y \to \infty$ respectively.

Next, transforming the ${}_4\phi_3$ on the right-hand side of (2.5) by (2.4) with $a = x$, $b = -q^{-n}/xy$, $c = y$, $e = q^{-n}/x$, $g = -xyq$, $h = q^{-n}/y$, we get a summation formula for terminating ${}_4\phi_3$ (cf. Andrews 1976, Th. 1)

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{-n}/x^2, y, -y; q; q \\ q^{-n}/x, -q^{-n}/x, y^2q \end{matrix}\right]
$$

$$
= \frac{[q; q]_n [x^2y^2q^2; q^2]_n [x^2q^2; q^2]_m [y^2q^2; q^2]_m}{[x^2q^2; q^2]_n [y^2q; q]_n [x^2y^2q^2; q^2]_m [q^2; q^2]_m} \qquad \text{...(2.12)}
$$

which corresponds to the following summation formula for the ordinary hypergeometric series:

$$
{}_3F_2\left[\begin{matrix}-n, -n-2x, y;\\ -n-x, 2y+1\end{matrix}\right] = \frac{(1)_n (1+x+y)_n (1+x)_m (1+y)_m}{(1+x)_n (1+2y)_n (1+x+y)_m (1)_m}. \quad \text{...(2.13)}
$$

In (2.12) letting $y \to 0$, we get q-analogue of a terminating version of a summation formula due to Mitra [Luke 1975, 272(10)]

$$
{}_3\phi_2\left[\begin{matrix}q^{-n}, \dfrac{q^{-n}}{x^2}, 0; q; q\\ \dfrac{q^{-n}}{x}, -\dfrac{q^{-n}}{x}\end{matrix}\right] = \frac{[q;q]_n [x^2q^2;q^2]_m}{[x^2q^2;q^2]_n [q^2;q^2]_m} \quad \text{...(2.14)}
$$

on the other hand (2.12) for $y \to \infty$ yields the following summation formula for terminating ${}_2\phi_2$

$$
{}_2\phi_2\left[\begin{matrix}q^{-n}, \dfrac{q^{-n}}{x^2}; q; -1\\ \dfrac{q^{-n}}{x}, -\dfrac{q^{-n}}{x}\end{matrix}\right] = \frac{[q;q]_n [x^2q^2;q^2]_m q^{n(n+1)/2} x^{2n-2m}}{[x^2q^2;q^2]_n [q^2;q^2]_m}. \quad \text{...(2.15)}
$$

Transforming the ${}_4\phi_3$ on the left-hand side of (2.12) by (2.4) with $a = y$, $b = -y$, $c = q^{-n}/x^2$, $e = y^2q$, $g = q^{-n}/x$, $h = -q^{-n}/x$, we get (which could also be obtained from (2.12) by writing the series in the reverse order)

$$
{}_4\phi_3\left[\begin{matrix}q^{-n}, q^{-n}/x^2, yq, -yq; q; q\\ q^{1-n}/x, -q^{1-n}/x, y^2q\end{matrix}\right]
$$

$$
= \frac{(-)^n [q;q]_n [x^2y^2q^2;q^2]_n [x^2q^2;q^2]_m [y^2q^2;q^2]_m}{q^n [x^2;q^2]_n [y^2q;q]_n [x^2y^2q^2;q^2]_m [q^2;q^2]_m} \quad \text{...(2.16)}
$$

which in the limit reduces to

$$
{}_3F_2\left[\begin{matrix}-n, -n-2x, 1+y;\\ 1-n-x, 2y+1\end{matrix}\right]
$$

$$
= \frac{(-)^n (1)_n (1+x+y)_n (1+x)_m (1+y)_m}{(x)_n (1+2y)_n (1+x+y)_m (1)_m}. \quad \text{...(2.17)}
$$

In (2.16) letting $y \to 0$, we get a q-analogue of a terminating version of another result of Mitra [Luke 1975, p. 272(9)]

$$
{}_3\phi_2\left[\begin{matrix}q^{-n}, q^{-n}/x^2, 0; q; q\\ q^{1-n}/x, -q^{1-n}/x\end{matrix}\right] = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m}{q^n [x^2;q^2]_n [q^2;q^2]_m} \quad \text{...(2.18)}
$$

on the other hand (2.16) for $y \to \infty$, yields

$$
{}_2\phi_2\left[\begin{matrix}q^{-n}, q^{-n}/x^2; q; -q^2\\ q^{1-n}/x, -q^{1-n}/x\end{matrix}\right] = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m q^{n(n-1)/2} x^{2n-2m}}{[x^2;q^2]_n [q^2;q^2]_m}. \quad \text{...(2.19)}
$$

Again transforming the ${}_4\phi_3$ on the left-hand side of (2.12) by (2.4) with $a = y$, $b = -y$, $c = \frac{q^{-n}}{x^2}$, $e = -\frac{q^{-n}}{x}$, $g = \frac{q^{-n}}{x}$, $h = y^2q$, we get

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, \frac{q^{-n}}{x^2}, \frac{q^{-n}}{xy}, -\frac{q^{-n}}{xy}; q; q \\ -\frac{q^{-n}}{x}, \frac{q^{1-n}}{x}, \frac{q^{-2n}}{x^2y^2} \end{matrix}\right]
$$

$$
= \frac{(-)^n x^n q^{n(n-1)/2} [q;q]_n [x^2y^2q;q]_n [x^2q^2;q^2]_m}{[x^2y^2q;q^2]_n [-xq;q]_n [x;q]_n [x^2y^2q^2;q^2]_m} \cdot \frac{[y^2q^2;q^2]_m}{[q^2;q^2]_m}, \qquad \text{...(2.20)}
$$

which on proceeding to the limit corresponds to (cf. Slater 1966, 2.3.3.12)

$$
{}_3F_2\left[\begin{matrix} -n, -n-2x, -n-x-y; \\ 1-n-x, -2n-2x-2y \end{matrix}\right]
= \frac{(-)^n (1)_n (1+2x+2y)_n (1+x)_m (1+y)_m}{2^{2n}(x)_n (\frac{1}{2}+x+y)_n (1+x+y)_m (1)_m}. \qquad \text{...(2.21)}
$$

Whereas, in (2.20) replacing x by $-x$ and then proceeding to the limit, we get (cf. Slater 1966, 2.3.3.12)

$$
{}_3F_2\left[\begin{matrix} -n, -n-2x, -n-x-y; \\ -n-x, -2n-2x-2y \end{matrix}\right]
= \frac{(1)_n (1+2x+2y)_n (1+x)_m (1+y)_m}{2^{2n}(1+x)_n (\frac{1}{2}+x+y)_n (1+x+y)_m (1)_m} \qquad \text{...(2.22)}
$$

In (2.20) letting $y \to \infty$, we get

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, \frac{q^{-n}}{x^2}, 0; q; q \\ -\frac{q^{-n}}{x}, \frac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n [q;q]_n [x^2q^2;q^2]_m x^{n-2m}}{[x;q]_n [-xq;q]_n [q^2;q^2]_m} \qquad \text{...(2.23)}
$$

while (2.20) for $y \to 0$, yields

$$
{}_2\phi_2\left[\begin{matrix} q^{-n}, \frac{q^{-n}}{x^2}; q; -q \\ -\frac{q^{-n}}{x}, \frac{q^{1-n}}{x} \end{matrix}\right] = \frac{(-)^n x^n q^{n(n-1)/2} [q;q]_n [x^2q^2;q^2]_m}{[x;q]_n [-xq;q]_n [q^2;q^2]_m}. \qquad \text{...(2.24)}
$$

(2.23) and (2.24) are q-analogues of terminating versions of two summation theorems for ${}_2F_1\,[\frac{1}{2}]$ due to Mitra [Luke 1975, p. 272 (9, 10)].

Lastly, writing terminating ${}_4\phi_3$ in (2.20) in the reverse order, we get

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, x^2y^2q^{1+n}, x, -xq; q; q \\ xyq, -xyq, x^2q \end{matrix}\right]
= \frac{x^n\,[q;q]_n\,[x^2q^2;q^2]_m\,[y^2q^2;q^2]_m}{[x^2q;q]_n\,[x^2y^2q^2;q^2]_m\,[q^2;q^2]_m} \qquad \ldots(2.25)
$$

which on proceeding to the limit becomes

$$
{}_3F_2\left[\begin{matrix} -n, 1+n+2x+2y, x; \\ 1+x+y, 1+2x \end{matrix}\right] = \frac{(1)_n\,(1+x)_m\,(1+y)_m}{(1+2x)_n\,(1+x+y)_m\,(1)_m}. \qquad \ldots(2.26)
$$

On the other hand in (2.25) replacing x by $-x$ and then proceeding to the limit in the usual way, we get

$$
{}_3F_2\left[\begin{matrix} -n, 1+n+2x+2y, 1+x; \\ 1+x+y, 1+2x \end{matrix}\right] = \frac{(-)^n\,(1)_n\,(1+x)_m\,(1+y)_m}{(1+2x)_n\,(1+x+y)_m\,(1)_m}. \qquad \ldots(2.27)
$$

Now, in (2.25) replacing y by y/x and then letting $x \to \infty$, we get a terminating version of q-analogue of Gauss' second summation theorem due to Andrews (1973, 1.8). Whereas in (2.25) replacing x by x/y and then letting $y \to \infty$, we get q-analogue of a terminating version of Gauss' second summation theorem different from the ones due to Andrews (1973, 1.8), viz.,

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, x^2q^{1+n}, 0; q; q \\ xq, -xq \end{matrix}\right]
= \begin{cases} 0 \text{ if } n \text{ is odd} \\ (-)^n\,q^{n(n+1)}x^{2n}\,\dfrac{[q;q^2]_n}{[x^2q^2;q^2]_n} \text{ if } n \text{ is even} \end{cases} \qquad \ldots(2.28)
$$

Next in (1.1) setting $y^2 = -aq$ and transforming the resulting terminating well-poised ${}_8\phi_7$ on the left-hand side by (2.1) with $b \to x$, $c \to -x$, $d \to b$, $e \to -q^{-n}$, we get a transformation between a Saalschützian ${}_4\phi_3$ on base q and a Saalschützian ${}_4\phi_3$ on the base q^2, viz,

$$
{}_4\phi_3\left[\begin{matrix} -\dfrac{aq}{x^2}, b, -q^{-n}, q^{-n}; q; q \\ -\dfrac{b}{a}\,q^{-2n}, \dfrac{aq}{x}, -\dfrac{aq}{x} \end{matrix}\right]
= \frac{[-aq^2;q^2]_n\,[-aq/x^2;q^2]_n\,[a^2q^2/b^2;q^2]_n}{[a^2q^2/x^2;q^2]_n\,[-aq/b;q]_{2n}} \times
$$

(*equation continued on p.* 1029)

$$\times\ {}_4\phi_3\begin{bmatrix} x^2, -\dfrac{aq}{b}, -\dfrac{aq^2}{b}, q^{-2n}; q^2; q^2 \\ -aq^2, \dfrac{a^2q^2}{b^2}, -\dfrac{x^2}{a}\, q^{1-2n} \end{bmatrix} \qquad \ldots(2.29)$$

In (2.29) replacing a by $-a$ and then setting $b = a$, we get q-analogue of a result of Bailey (1953, 2.2) [a terminating version of Watson's summation theorem which does not follows directly from the Watson's summation theorem for ${}_3F_2(+1)$. See also Jain (1980a) for alternative proof] :

$${}_4\phi_3\begin{bmatrix} \dfrac{aq}{x^2}, a, -q^{-n}, q^{-n}; q; q \\ \dfrac{aq}{x}, -\dfrac{aq}{x}, q^{-2n} \end{bmatrix} = \frac{[aq; q^2]_n\, [aq^2/x^2; q^2]_n}{[q^2; q^2]_n\, [a^2q^2/x^2; q^2]_n}. \qquad \ldots(2.30)$$

Furthermore (1.1) for $y = aq^{1+n}$, yields

$${}_5\phi_4\begin{bmatrix} x^2, a^2q^{2+2n}, -\dfrac{aq}{b} -\dfrac{aq^2}{b}, q^{-2n}; q^2; q^2 \\ -aq, -aq^2, \dfrac{a^2q^2}{b^2}, x^2q^2 \end{bmatrix}$$

$$= \frac{x^{2n}\, [a^2q^2/x^2; q^2]_n\, [q^2; q^2]_n}{[a^2q^2; q^2]_n\, [x^2q^2; q^2]_n}$$

$$\times\ {}_6\phi_5\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, x, -x, b; q; -\dfrac{aq}{bx^2} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{x}, -\dfrac{aq}{x}, \dfrac{aq}{b} \end{bmatrix} \text{to } (n+1) \text{ terms}$$

$$\ldots(2.31)$$

Now, in (2.31) setting $a = -bx^2$ and summing the truncated ${}_6\phi_5$ on the right hand side by a result of Agarwal (1953, p. 444), we get q-analogue of a result of Bailey [1929, p. 512(C)] [see Verma and Jain (1980, 5.3) for alternative proof]

$${}_4\phi_3\begin{bmatrix} b^2x^4q^{2+2n}, x^2, x^2q, q^{-2n}; q^2; q^2 \\ bx^2q, bx^2q^2, x^4q^2 \end{bmatrix} = \frac{x^{2n}\, [-q; q]_n\, [bq; q]_n}{[-x^2q; q]_n\, [bx^2q; q]_n}. \qquad \ldots(2.32)$$

On the other hand in (2.31) replacing a by $-a$ and then setting $x^2 = aq$, $b = aq/c$ and summing the resulting truncated ${}_4\phi_3$ on the right-hand side by another result of Agarwal (1953, p. 444) we get a result of Andrews (1979, 4.3) which is q-analogue of a result of Carlitz (1963) [this result was obtained later on independently by Cvetković and Simić (1973) and Gasper (1975); see also Verma and Jain (1980, 5.4) where it is shown that the result of Andrews (1979, 4.3) may be obtained from (2.32) by using (2.4)].

However, multiplying (2.3) by

$$\frac{[c;q]_{2n}\,[x;q]_{2n}\,[y;q]_{2n}\,z^{2n}}{[q;q]_{2n}\,[d;q]_{2n}\,[-xy;q]_{2n}}$$

and summing with respect to n from 0 to ∞, we get

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[c;q]_{n+r}\,[x;q]_r\,[x;q]_n\,[y;q]_r\,[y;q]_n\,(-)^r\,z^{n+r}}{[q;q]_r\,[q;q]_n\,[d;q]_{n+r}\,[-xy;q]_r\,[-xy;q]_n}$$

$$= {}_6\phi_5\left[\begin{matrix}x^2,\, y^2,\, xy,\, xyq,\, c,\, cq;\, q^2;\, z^2\\ x^2y^2,\, -xy,\, -xyq,\, d,\, dq\end{matrix}\right]. \qquad \text{...(2.33)}$$

Indeed, (2.33) for $c \to 0$, yields a q-analogue of a result of Bailey (1953, 4.3), whereas in (2.33) letting $y \to \infty$, we get

$$\phi^{(1)}\left[c;\, x,\, x;\, d;\, \frac{z}{x},\, -\frac{z}{x};\, q\right] = {}_3\phi_2\left[\begin{matrix}x^2,\, c,\, cq;\, q^2;\, \dfrac{z^2}{x^2}\\ d,\, dq\end{matrix}\right], \qquad \text{...(2.34)}$$

[where $\phi^{(1)}$ is one of the basic Appell functions defined by Jackson (1942)] Now using the reducibility of $\phi^{(1)}$ due to Andrews (1972, Th. 1), we get

$${}_3\phi_2\left[\begin{matrix}\dfrac{d}{c},\, \dfrac{z}{x},\, \dfrac{-z}{x};\, q;\, c\\ z,\, -z\end{matrix}\right] = \frac{[d;q]_\infty[z^2/x^2;q^2]_\infty}{[c;q]_\infty\,[z^2;q^2]_\infty}\,{}_3\phi_2\left[\begin{matrix}x^2,\, c,\, cq;\, q^2;\, \dfrac{z^2}{x^2}\\ d,\, dq\end{matrix}\right].$$

Furthermore, in (2.33) setting $c = d$, we get

$${}_2\phi_1\left[\begin{matrix}x,\, y;\, q;\, -z\\ -xy\end{matrix}\right]{}_2\phi_1\left[\begin{matrix}x,\, y;\, q;\, z\\ -xy\end{matrix}\right] = {}_4\phi_3\left[\begin{matrix}x^2,\, y^2,\, xy,\, xyq;\, q^2;\, z^2\\ x^2y^2,\, -xy,\, -xyq\end{matrix}\right]. \qquad \text{...(2.35)}$$

In (2.35) replacing z by $z(1-q)^{-1}$ and setting x or y is of the form q^{-N}, then proceeding to the limit, we get a result of Bailey (1928, 2.07), while in (2.35) replacing y, z by $-y$ and $z(1-q)$ respectively and then proceeding to the limit, we get another result of Bailey (1928, 2.09).

On the other hand multiplying both of the sides of (2.5) by

$$\frac{[c;q]_n\,[xq;q]_n\,[yq;q]_n\,z^n}{[q;q]_n\,[d;q]_n\,[-xyq;q]_n}$$

and summing with respect to n from 0 to ∞, we get

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[c;q]_{n+r}\,[x;q]_r\,[y;q]_r\,[xq;q]_n\,[yq;q]_n\,(-q)^r\,z^{n+r}}{[q;q]_r\,[q;q]_n\,[d;q]_{n+r}\,[-xyq;q]_r\,[-xyq;q]_n} =$$

(*equation continued on p.* 1031)

$$= {}_6\phi_5\begin{bmatrix} c,\ cq,\ xyq,\ xyq^2,\ x^2q^2,\ y^2q^2;\ q^2;\ z^2 \\ d,\ dq,\ -xyq,\ -xyq^2,\ x^2y^2q^2 \end{bmatrix} + \frac{(1-c)(1-xyq)\,z}{(1-d)(1+xyq)}$$

$$\times\ {}_6\phi_5\begin{bmatrix} cq,\ cq^2,\ xyq^2,\ xyq^3,\ x^2q^2,\ y^2q^2;\ q^2;\ z^2 \\ dq,\ dq^2,\ -xyq^2,\ -xyq^3,\ x^2y^2q^2 \end{bmatrix}. \qquad \ldots(2.36)$$

(2.36) for $c = d$, yields

$${}_2\phi_1\begin{bmatrix} x,\ y;\ q;\ -zq \\ -xyq \end{bmatrix} {}_2\phi_1\begin{bmatrix} xq,\ yq;\ q;\ z \\ -xyq \end{bmatrix}$$

$$= {}_4\phi_3\begin{bmatrix} xyq,\ xyq^2,\ x^2q^2,\ y^2q^2;\ q^2;\ z^2 \\ -xyq,\ -xyq^2,\ x^2y^2q^2 \end{bmatrix}$$

$$+ \frac{(1-xyq)\,z}{(1+xyq)}\ {}_4\phi_3\begin{bmatrix} xyq^2,\ xyq^3,\ x^2q^2,\ y^2q^2;\ q^2;\ z^2 \\ -xyq^2,\ -xyq^3,\ x^2y^2q^2 \end{bmatrix}. \qquad \ldots(2.37)$$

In (2.37) replacing z by $z(1-q)^{-1}$ and proceeding to the limit, we get

$${}_2F_0\begin{bmatrix} x,\ y;\ -z \\ - \end{bmatrix} {}_2F_0\begin{bmatrix} x+1,\ y+1;\ z \\ - \end{bmatrix}$$

$$= {}_4F_1\begin{bmatrix} 1+x,\ 1+y,\ \tfrac{1}{2}(1+x+y),\ \tfrac{1}{2}(2+x+y);\ z^2 \\ 1+x+y \end{bmatrix} + \frac{(1+x+y)\,z}{2}$$

$$\times\ {}_4F_1\begin{bmatrix} 1+x,\ 1+y,\ \tfrac{1}{2}(2+x+y),\ \tfrac{1}{2}(3+x+y);\ z^2 \\ 1+x+y \end{bmatrix} \qquad \ldots(2.38)$$

provided either x or y is a non-negative integer otherwise the series are divergent. Whereas in (2.37) replacing y, z by $-y$ and $z(1-q)$ respectively and then proceeding to the limit, we get

$${}_1F_1\begin{bmatrix} x;\ -z \\ 1+x+y \end{bmatrix} {}_1F_1\begin{bmatrix} 1+x;\ z \\ 1+x+y \end{bmatrix}$$

$$= {}_2F_3\begin{bmatrix} 1+x,\ 1+y;\ z^2 \\ 1+x+y,\ \tfrac{1}{2}(1+x+y),\ \tfrac{1}{2}(2+x+y) \end{bmatrix} + \frac{2z}{(1+x+y)}$$

$$\times\ {}_2F_3\begin{bmatrix} 1+x,\ 1+y;\ z^2 \\ 1+x+y,\ \tfrac{1}{2}(2+x+y),\ \tfrac{1}{2}(3+x+y) \end{bmatrix} \qquad \ldots(2.39)$$

(2.36) on letting $y \to \infty$, yields

$$\phi^{(1)}\left[c;\ x,\ xq;\ d;\ \frac{z}{x},\ -\frac{z}{x};\ q\right] = {}_3\phi_2\begin{bmatrix} c,\ cq,\ x^2q^2;\ q^2;\ \dfrac{z^2}{x^2} \\ d,\ dq \end{bmatrix}$$

$$- \frac{(1-c)\,z}{(1-d)}\ {}_3\phi_2\begin{bmatrix} cq,\ cq^2,\ x^2q^2;\ q^2;\ \dfrac{z^2}{x^2} \\ dq,\ dq^2 \end{bmatrix} \qquad \ldots(2.40)$$

which on using the reducibility of $\phi^{(1)}$ due to Andrews (1972, Th. 1), becomes

$$\frac{[c;q]_\infty [z;q]_\infty [-zq;q]_\infty}{[d;q]_\infty \left[\frac{z^2}{x^2};q^2\right]_\infty} {}_3\phi_2\left[\begin{matrix}\frac{d}{c}, \frac{z}{x}, -\frac{z}{x}; q; q\\ z, -zq\end{matrix}\right]$$

$$= {}_3\phi_2\left[\begin{matrix}c, cq, x^2q^2; q^2; \frac{z^2}{x^2}\\ d, dq\end{matrix}\right]$$

$$-\frac{(1-c)\,z}{(1-d)}\, {}_3\phi_2\left[\begin{matrix}cq, cq^2, x^2q^2; q^2; \frac{z^2}{x^2}\\ dq, dq^2\end{matrix}\right]. \qquad ...(2.41)$$

Lastly, in (2.29) replacing a and x by $-aq^{-2n}$ and xq^{-n} respectively and then transforming the ${}_4\phi_3$ on the right hand side by (2.4) with, first replacing q by q^2 and then setting $a \to x^2q^{-2n}$, $b \to aq^{1-2n}/b$, $c \to aq^{2-2n}/b$, $e \to aq^{2-2n}$, $g \to x^2q^{1-2n}/a$, $h \to a^2q^{2-4n}/b^2$, we get a result of Singh (1959) viz.,

$${}_4\phi_3\left[\begin{matrix}\frac{aq}{x^2}, b, -q^{-n}, q^{-n}; q; q\\ \frac{b}{a}, \frac{aq^{1-n}}{x}, -\frac{aq^{1-n}}{x}\end{matrix}\right] = \frac{\left[\frac{1}{a};q^2\right]_n \left[\frac{bq^2}{a};q^2\right]_n \left[\frac{bx^2}{qa^2};q^2\right]_n}{\left[\frac{b}{a};q\right]_{2n}\left[\frac{x^2}{a^2};q^2\right]_n}$$

$$\times\, {}_4\phi_3\left[\begin{matrix}\frac{aq^2}{x^2}, bq, \frac{aq^{2-2n}}{b}, q^{-2n}; q^2; q^2\\ aq^{2-2n}, \frac{b}{a}q^2, \frac{a^2q^{3-2n}}{bx^2}\end{matrix}\right]. \qquad ...(2.42)$$

Using (2.42) Singh (1959) obtained q-analogue of an identity of Cayley-Orr type. The other identities from which Singh obtained q-analogue of identities of Cayley-Orr type could be obtained by transforming ${}_4\phi_3$'s in (2.42) by using (2.29) and (2.4).

§3. In (1.2) setting $y = aq^{1+n}$, we get

$${}_5\phi_4\left[\begin{matrix}x, aq^{1+n}, \left(\frac{aq}{b}\right)^{1/2}, -\left(\frac{aq}{b}\right)^{1/2}, q^{-n}; q; q\\ (aq)^{1/2}, -(aq)^{1/2}, \frac{aq}{b}, xq\end{matrix}\right]$$

$$= \frac{x^n [q;q]_n \left[\frac{aq}{x};q\right]_n}{[aq;q]_n [xq;q]_n}\, {}_6\phi_5\left[\begin{matrix}a, q^2\sqrt{a}, -q^2\sqrt{a}, b, x, xq; q^2; \frac{aq}{bx^2}\\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{b}, \frac{aq^2}{x}, \frac{aq}{x}\end{matrix}\right]$$

$$\text{to } (m+1) \text{ terms} \qquad ...(3.1)$$

where m is the greatest integer $\leqslant \frac{n}{2}$.

Special Cases

(i) In (3.1) setting $a = bx^2q$ and then summing the truncated ${}_6\phi_5$ on the right-hand side by a result of Agarwal (1953, p. 444), we get the summation theorem

$$
{}_4\phi_3\left[\begin{matrix} x,\ -xq,\ bx^2\,q^{2+n},\ q^{-n};\ q;\ q \\ x^2q^2,\ xq\sqrt{b},\ -xq\sqrt{b} \end{matrix}\right]
$$

$$
= \frac{x^n\,[q;q]_n\,[bxq^2;q]_n}{[xq;q]_n\,[bx^2q^2;q]_n}\cdot\frac{[bx^2q^3;q^2]_m\,[bq^2;q^2]_m\,[xq^2;q]_{2m}}{[q^2;q^2]_m\,[x^2q^3;q^2]_m\,[bxq^2;q]_{2m}}. \qquad \text{...(3.2)}
$$

Now, in (3.2) proceeding to the limit, we get the sum of a terminating ${}_3F_2$ very similar to a terminating version of the Watson's summation formula for ${}_3F_2$:

$$
{}_3F_2\left[\begin{matrix} -n,\ 2+b+n+2x,\ x; \\ 1+\frac{1}{2}b+x,\ 2x+2 \end{matrix}\right]
$$

$$
= \frac{(1)_n\,(2+b+x)_n\,(\frac{3}{2}+x+\frac{1}{2}b)_m\,(1+\frac{1}{2}b)_m\,(1+\frac{1}{2}x)_{2m}}{(1+x)_n\,(2+b+2x)_n\,(1)_m\,(\frac{3}{2}+x)_m\,(1+\frac{1}{2}b+\frac{1}{2}x)_{2m}}. \qquad \text{...(3.3)}
$$

On the other hand in (3.2) replacing x by $-x$ and then proceeding to the limit, we get

$$
{}_3F_2\left[\begin{matrix} -n,\ 2+b+n+2x,\ 1+x; \\ 1+\frac{1}{2}b+x,\ 2+2x \end{matrix}\right]
$$

$$
= \frac{(-)^n\,(1)_n\,(\frac{3}{2}+\frac{1}{2}b+x)_m\,(1+\frac{1}{2}b)_m}{(2+b+2x)_n\,(1)_m\,(\frac{3}{2}+x)_m}. \qquad \text{...(3.4)}
$$

(ii) In (3.1) setting $b = aq^2/x$ and then summing the resulting truncated ${}_4\phi_3$ on the right-hand side by a result of Agarwal (1953, p. 444), we get

$$
{}_5\phi_4\left[\begin{matrix} x,\ aq^{1+n},\ \left(\frac{x}{q}\right)^{1/2},\ -\left(\frac{x}{q}\right)^{1/2},\ q^{-n};\ q;\ q \\ (aq)^{1/2},\ -(aq)^{1/2},\ \frac{x}{q},\ xq \end{matrix}\right]
$$

$$
= \frac{x^{n-m}\left[\frac{aq}{x};q\right]_n [q;q]_n\,[aq^2;q^2]_m\,[xq;q^2]_m}{q^m\,[aq;q]_n\,[xq;q]_n\,[q^2;q^2]_m\left[\frac{aq}{x};q^2\right]_m}. \qquad \text{...(3.5)}
$$

(3.5) on proceeding to the limit, reduces to

$$
{}_4F_3\left[\begin{matrix} x,\ \frac{1}{2}x-\frac{1}{2},\ 1+a+n,\ -n; \\ x-1,\ x+1,\ \frac{1}{2}+\frac{1}{2}a \end{matrix}\right]
$$

$$
= \frac{(1)_n\,(1+a-x)_n\,(1+\frac{1}{2}a)_m\,(\frac{1}{2}+\frac{1}{2}a)_m}{(1+a)_n\,(1+x)_n\,(1)_m\,(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}x)_m}. \qquad \text{...(3.6)}
$$

Next, in (1.2) replacing b and x by aq/b and aq/x respectively and then letting $a \to 0$, we get q-analogue of a result of Bailey (1928; 5.41) (Jain 1980a for alternative proof), viz.,

$$
{}_4\phi_3\left[\begin{matrix} y, yq, q^{-n}, q^{-n+1}; q^2; \dfrac{bx^2q^{2n}}{y^2} \\ bq, x, xq \end{matrix}\right]
$$

$$
= \frac{\left[\dfrac{x}{y}; q\right]_n}{[x; q]_n} {}_4\phi_3\left[\begin{matrix} y, \sqrt{b}, -\sqrt{b}, q^{-n}; q; q \\ b, \dfrac{y}{x} q^{1-n}, 0 \end{matrix}\right]. \qquad \ldots(3.7)
$$

Lastly, we prove the following transformation (cf. Jain 1980b; 1.2)

$$
{}_4\phi_3\left[\begin{matrix} c, d, x, -xq; q; \dfrac{q^2}{cd} \\ \sqrt{q}, -\sqrt{q}, x^2q^2 \end{matrix}\right] = \frac{\left[\dfrac{q^2}{c}; q\right]_\infty \left[\dfrac{q^2}{d}; q\right]_\infty}{[q^2; q]_\infty \left[\dfrac{q^2}{cd}; q\right]_\infty}
$$

$$
\times {}_8\psi_8\left[\begin{matrix} c, cq, d, dq, \dfrac{q}{x^2}, xq^3, q^{5/2}, -q^{5/2}; q^2; \dfrac{x^2q^5}{c^2d^2} \\ \dfrac{q^3}{c}, \dfrac{q^2}{c}, \dfrac{q^3}{d}, \dfrac{q^2}{d}, x^2q^3, xq, q^{1/2}, -q^{1/2}; 2 \end{matrix}\right] \qquad \ldots(3.8)
$$

provided $c, d \neq q$, $|q^2/cd| < 1$.

Proof of (3.8) — In (3.2) replacing b by $b/(x^2q^2)$ and then multiplying on both the sides by $\dfrac{[b; q]_n [bq^2; q^2]_n [c; q]_n [d; q]_n [q^{-N}; q]_n}{[q; q]_n [b; q^2]_n \left[\dfrac{bq}{c}; q\right]_n \left[\dfrac{bq}{d}; q\right]_n [bq^{1+N}, q]_n} \left(\dfrac{bq^{1+N}}{cd}\right)^n$ and summing with respect to n from 0 to N, we get on some simplification

$$
\frac{[bq; q]_N \left[\dfrac{bq}{cd}; q\right]_N}{\left[\dfrac{bq}{c}; q\right]_N \left[\dfrac{bq}{d}; q\right]_N} {}_5\phi_4\left[\begin{matrix} c, d, x, -xq, q^{-N}; q; q \\ \sqrt{b}, -\sqrt{b}, x^2 q^2, \dfrac{cd}{b} q^{-N} \end{matrix}\right]
$$

$$
= \sum_{n=0}^{[N/2]} \frac{[c; q]_{2n} [d; q]_{2n} [q^{-N}; q]_{2n} [bq^2; q^2]_{2n} \left[\dfrac{b}{x}; q\right]_{2n} [bq; q^2]_n}{\left[\dfrac{bq}{c}; q\right]_{2n} \left[\dfrac{bq}{d}; q\right]_{2n} [bq^{1+N}; q]_{2n} [b; q^2]_{2n} [xq; q]_{2n} [q^2; q^2]_n}
$$

$$
\times \frac{\left[\dfrac{b}{x^2}; q^2\right]_n [xq^2; q]_{2n}}{[x^2q^3; q^2]_n \left[\dfrac{b}{x}; q\right]_{2n}} \left(\frac{bxq^{1+N}}{cd}\right)^{2n} +
$$

(*equation continued on p.* 1035)

$$+\sum_{n=0}^{[N/2]} \frac{[c;q]_{2n+1}[d;q]_{2n+1}[q^{-N};q]_{2n+1}[bq^2;q^2]_{2n+1}}{\left[\frac{bq}{c};q\right]_{2n+1}\left[\frac{bq}{d};q\right]_{2n+1}[bq^{1+N};q]_{2n+1}}$$

$$\times \frac{\left[\frac{b}{x};q\right]_{2n+1}[bq;q^2]_n\left[\frac{b}{x^2};q^2\right]_n[xq^2;q]_{2n}}{[b;q^2]_{2n+1}[xq;q]_{2n+1}[q^2;q^2]_n[x^2q^3;q^2]_n\left[\frac{b}{x};q\right]_{2n}}\left(\frac{bxq^{1+N}}{cd}\right)^{2n+1}. \quad \ldots(3.9)$$

In (3.9) letting $N \to \infty$ and then setting $b = q$ and $n = -r - 1$ in the last expression on the right-hand side and then combining the two series we get (3.8) (provided $c, d \neq q$)

§4. In (1.3) setting $y = aq^n$, we get a summation formula for terminating Saalschützian ${}_5\phi_4$ due to Andrews (1979, 4.5). Whereas in (1.3) on setting $y = aq^{1+n}$, we get the following transformation

$${}_6\phi_5\left[\begin{matrix}aq, aq^2, aq^3, x^3, a^3q^{3+3n}, q^{-3n}; q^3; q^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3, x^3q^3\end{matrix}\right]$$

$$= \frac{x^{3n}[q^3;q^3]_n\left[\frac{a^3q^3}{x^3};q^3\right]_n}{[a^3q^3;q^3]_n[x^3q^3;q^3]_n}$$

$$\times {}_6\phi_5\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, x, wx, wx^2; q; \frac{aq}{x} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{x}, \frac{aqw^2}{x}, \frac{aqw}{x}\end{matrix}\right] \text{to } (n+1) \text{ terms} \quad \ldots(4.1)$$

In (4.1) setting $x^3 = a$ and summing the truncated ${}_6\phi_5$ on the right-hand side by a result of Agarwal (1953, p. 444), we get a summation formula for terminating Saalschützian ${}_5\phi_4$ different from a result of Andrews (1979, 4.5), viz.,

$${}_5\phi_4\left[\begin{matrix}a, aq, aq^2\ a^3q^{3+3n}, q^{-3n}; q^3; q^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^3, -a^{3/2}q^3\end{matrix}\right] = \frac{a^n[q^3;q^3]_n[aq;q]_n}{[a^3q^3;q^3]_n[q;q]_n}. \quad \ldots(4.2)$$

Similarly, in (1.4) setting $y = aq^n$, we get a summation theorem for terminating Saalschützian ${}_5\phi_4$ due to Adrews (1979, 4.7). On the other hand, in (1.4) setting $y = aq^{1+n}$, we get a transformation

$${}_6\phi_5\left[\begin{matrix}a^{1/3}, wa^{1/3}, w^2a^{1/3}, x, aq^{1+n}, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xq\end{matrix}\right] =$$

(*equation continued on p.* 1036)

$$= \frac{x^n \left[\frac{aq}{x}; q\right]_n [q; q]_n}{[aq; q]_n [xq; q]_n}$$

$$\times {}_6\phi_5 \left[\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2; q^3; \frac{a}{x^3} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^3}{x}, \frac{aq^2}{x}, \frac{aq}{x} \end{matrix}\right] \text{ to } (m+1) \text{ terms} \quad \ldots(4.3)$$

(where m is the greatest integer $\leqslant n/3$)

Special Cases

(i) In (4.3) setting $a = x^3q^3$ and then summing the truncated ${}_6\phi_5$ on the right-hand side by a formula of Agarwal (1953, p. 444), we get sum of a terminating Saalschützian ${}_5\phi_4$:

$${}_5\phi_4 \left[\begin{matrix} x, wxq, w^2xq, x^3q^{4+n}, q^{-n}; q; q \\ (xq)^{3/2}, -(xq)^{3/2}, x^{3/2}q^2, -x^{3/2}q^2 \end{matrix}\right]$$

$$= \frac{x^n [x^2q^4; q]_n [q; q]_n [x^3q^6; q^3]_m [xq^3; q]_{3m}}{[x^3q^4; q]_n [xq; q]_n [q^3; q^3]_m [x^2q^4; q]_{3m}}. \quad \ldots(4.4)$$

In (4.4) proceeding to the limit in a usual way, we get (1.5).

(ii) In (4.3) setting $x = \sqrt{a}$ and then summing the resulting truncated ${}_4\phi_3$ on the right hand side by a formula of Agarwal (1953; p. 444), we get another summation formula for terminating Saalschützian ${}_5\phi_4$:

$${}_5\phi_4 \left[\begin{matrix} a^{1/3}, wa^{1/3}, w^2a^{1/3}, aq^{1+n}, q^{-n}; q; q \\ q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}\right] = \frac{(\sqrt{a})^{n-m} [q; q]_n [aq^3; q^3]_m}{[aq; q]_n [q^3; q^3]_m}. \quad \ldots(4.5)$$

(4.5) as a limiting case yields

$${}_3F_2 \left[\begin{matrix} \frac{a}{3}, 1+a+n, -n; \frac{3}{4} \\ \frac{1}{2}+\frac{a}{2}, 1+\frac{a}{2} \end{matrix}\right] = \frac{(1)_n \left(1+\frac{a}{3}\right)_m}{(1+a)_n (1)_m}. \quad \ldots(4.6)$$

On the other hand in (4.5) replacing $\sqrt{a}$ by $-\sqrt{a}$ and then proceeding to the limit, we get

$${}_3F_2 \left[\begin{matrix} \frac{a}{3}, 1+a+n, -n; \frac{3}{4} \\ \frac{a}{2}, \frac{1}{2}+\frac{a}{2} \end{matrix}\right] = \frac{(-)^{n-m} (1)_n \left(1+\frac{a}{3}\right)_m}{(1+a)_n (1)_m}. \quad \ldots(4.7)$$

(iii) Lastly, in (4.3) setting $x = q\sqrt{a}$ and then summing the resulting truncated ${}_4\phi_3$ on the right-hand side by a result of Agarwal (1953, p. 444), we get sum of a terminating Saalschützian ${}_6\phi_5$:

$$
{}_6\phi_5\left[\begin{matrix} a^{1/3}, wa^{1/3}, w^2a^{1/3}, q\sqrt{a}, aq^{1+n}, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{matrix}\right]
$$

$$
= \frac{[q;q]_n\,[\sqrt{a};q]_n\,[aq^3;q^3]_m\,[q^6\sqrt{a};q^3]_m\,(\sqrt{a})^{n-m}}{[aq;q]_n\,[q^2\sqrt{a};q]_n\,[q^3;q^3]_m\,[\sqrt{a};q^3]_m}. \qquad \ldots(4.8)
$$

However, in (4.8) proceeding to the limit, we get

$$
{}_4F_3\left[\begin{matrix} \frac{a}{3}, 1+\frac{a}{2}, 1+a+n, -n; \frac{3}{4} \\ \frac{a}{2}, \frac{1}{2}+\frac{a}{2}, 2+\frac{a}{2} \end{matrix}\right]
$$

$$
= \frac{(1)_n\left(\frac{a}{2}\right)_n\left(1+\frac{a}{3}\right)_m\left(2+\frac{a}{6}\right)_m}{(1+a)_n\left(2+\frac{a}{2}\right)_n(1)_m\left(\frac{a}{6}\right)_m} \qquad \ldots(4.9)
$$

whereas, in (4.8) replacing $\sqrt{a}$ by $-\sqrt{a}$ and then proceeding to the limit, we get (4.7).

References

Agarwal, R. P. (1953). On the partial sums of series of hypergeometric type. *Proc. Camb. phil. Soc.*, **49**, 441–45.

Anderson, E. S. (1953). Two summation formulae for product sums of binomial coefficients. *Math. Scand.*, **1**, 261–62.

Andrews, G. E. (1972). Summations and transformations for basic Appell series. *J. Lond. math. Soc.* (2), **4**, 618–22.

———— (1973). On the q-analogue of Kummer's theorem and applications. *Duke Math. J.*, **40**, 525–28.

———— (1976). On q-analogues of the Watson and Whipple summations. *SIAM. J. Math. Analysis*, **7**, 332–36.

———— (1979). Connection coefficients problems and partitions. *Proc. Symp. Pure Math.*, **34**, 1–24.

Bailey, W. N. (1928). Products of generalised hypergeometric series. *Proc. Lond. math. Soc.* (2), **28**, 242–54.

———— (1929). Some identities involving generalised hypergeometric series. *Proc. Lond. math. Soc.* (2), **29**, 503–16.

———— (1932). On one of Ramanujan's theorems. *J. Lond. math. Soc.*, **7**, 34–36.

———— (1935). Generalised Hypergeometric Series. Cambridge Tract.

———— (1950). On the analogue of Dixon's theorem for bilateral basic hypergeometric series. *Q. Jl Math.* (Oxford), (2), **1**, 318–20.

———— (1953). On the sum of a terminating ${}_3F_2$ (1). *Q. Jl Math.* (Oxford) (2), **4**, 237–40.

Carlitz, L. (1963). Summation of a special ${}_4F_3$. *Boll. Unione Math. Ital.* (3), **8**, 90–93.

———— (1969). Some formulas of F. H. Jackson. *Monatesheft Math.*, **73**, 193–98.

Cvetković, D. M., and Simić, S. M. (1973). On enumeration of certain types of sequences. *Publ. Elektrotehn. Fak. Univ. Beograd, Ser. Mat. Fiz.*, No. 412–460, 159–64.

Gasper, G. (1975). Positive integrals of Bessel functions. *SIAM J. Math. Analysis*, **6**, 868–81.

Gould, H. W. (1961). The q-series generalization of a formula of sparre Andersen. *Math. Scand.*, **9**, 90–94.

Jackson, F. H. (1941). Certain, q-identities. *Q. Jl Math.* (Oxford), **12**, 167–72.

———— (1942). On basic double hypergeometric functions. *Q. Jl Math.* (Oxford), **13**, 69–82.

Jain, V. K. (1980a). Some transformations of basic hypergeometric series. *SIAM J. Math. Analysis*, (to appear).

———— (1980b). Some transformations of basic hypergeometric series and their applications. *Proc. Am. math. Soc.*, **78**, 375–84.

Luke, Y. L. (1975). Mathematical Functions and Their Approximations. Academic Press, New York.

Sears, D. B. (1951). On the transformation theory of basic hypergeometric functions. *Proc. Lond. math. Soc.* (2), **53**, 158–80.

Singh, V. N. (1959). The basic analogues of identities of Cayley-orr type. *J. Lond. math. Soc.*, **34**, 15–22.

Slater, L. J. (1966). Generalised Hypergeometric Series. Cambridge University Press, Cambridge.

Verma, A., and Jain, V. K. (1980). Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type. *J. Math. Anal. Appl.*, (to appear).