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ON SOME NEW INTEGRAL INEQUALITIES INVOLVING FUNCTIONS AND THEIR DERIVATIVES

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The present paper deals with some new integral inequalities involving functions and their derivatives which may be regarded as further generalizations of the well-known integral inequalities of Gronwall (1919) and Bihari (1956).

1. INTRODUCTION

In recent years, inequalities are playing a very significant role in all fields of mathematics, and present a very active and attractive field of research. As examples, let us cite the fields of ordinary and partial differential equations, which are dominated by inequalities and variational principles involving functions and their derivatives. One of the most useful techniques used in the theory of ordinary and partial differential equations and integral equations consists in applying so-called Gronwall type inequalities (see References at the end). The purpose of this paper is to establish some new integral inequalities involving functions and their derivatives which in the special cases contains the well-known integral inequalities of Gronwall (1919) and Bihari (1956) in the literature.

2. MAIN RESULTS

In this section we state and prove our main results on integral inequalities involving functions and their derivatives.

A useful integral inequality is embodied in the following theorem.

Theorem 1 — Let $x(t)$, $\dot{x}(t)$, $a(t)$ and $b(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$ and $x(0) = 0$, for which the inequality

$$x(t)\,\dot{x}(t) \leqslant k + \int_0^t a(s)\,x(s)\,(x(s) + \dot{x}(s))\,ds + \int_0^t b(s)\,x(s)\,\dot{x}(s)\,ds \qquad \ldots(1)$$

holds for all $t \in I$, where k is a nonnegative constant. Then

$$x(t) \leqslant \Big[2k \int_0^t \Big\{\exp\Big(\int_0^s b(\tau)\,d\tau\Big) + \int_0^s 2a(\tau)\exp\Big(\int_0^\tau [1 + 2a(\xi) + b(\xi)]\,d\xi\Big) \times \exp\Big(\int_\tau^s b(\xi)\,d\xi\Big)\Big\}\,ds\Big]^{1/2} \qquad \ldots(2)$$

for all $t \in I$.

PROOF : Define

$$m(t) = k + \int_0^t a(s)\, x(s)\, (x(s) + \dot{x}(s))\, ds + \int_0^t b(s)\, x(s)\, \dot{x}(s)\, ds, \; m(0) = k \qquad ...(3)$$

then (1) can be restated as

$$x(t)\, \dot{x}(t) \leqslant m(t). \qquad ...(4)$$

Integrating both sides of (4) from 0 to t we have

$$x^2(t) \leqslant 2 \int_0^t m(s)\, ds. \qquad ...(5)$$

Differentiating (3) and using (4) and (5) we have

$$\dot{m}(t) \leqslant 2a(t) \int_0^t m(s)\, ds + a(t)\, m(t) + b(t)\, m(t). \qquad ...(6)$$

From (6) we see that

$$\dot{m}(t) \leqslant 2a(t) \left[m(t) + \int_0^t m(s)\, ds \right] + b(t)\, m(t). \qquad ...(7)$$

If we put

$$v(t) = m(t) + \int_0^t m(s)\, ds, \quad v(0) = m(0) = k, \qquad ...(8)$$

then it follows by differentiating (8) and using the facts that

$$\dot{m}(t) \leqslant 2a(t)\, v(t) + b(t)\, m(t)$$

from (7) and $m(t) \leqslant v(t)$ from (8) we see that the inequality

$$\dot{v}(t) \leqslant [1 + 2a(t) + b(t)]\, v(t)$$

is satisfied, which implies the estimation for $v(t)$ such that

$$v(t) \leqslant k \exp \left(\int_0^t [1 + 2a(s) + b(s)]\, ds \right).$$

Substituting this bound on $v(t)$ in (7) we have

$$\dot{m}(t) \leqslant b(t)\, m(t) + 2ka(t) \exp \left(\int_0^t [1 + 2a(s) + b(s)]\, ds \right),$$

for all $t \in I$, which implies the estimate for $m(t)$ such that

$$m(t) \leqslant k \left[\exp\left(\int_0^t b(s)\, ds \right) + \int_0^t 2a(s) \exp\left(\int_0^s [1 + 2a(\tau) + b(\tau)]\, d\tau \right) \right.$$
$$\left. \times \exp\left(\int_s^t b(\tau)\, d\tau \right) ds \right]. \qquad \ldots(9)$$

Now, substituting this bound on $m(t)$ in (4) and integrating both sides from 0 to t we obtain the desired bound in (2).

We note that the integral inequality established in Theorem 1 may be regarded as a further generalization of the well-known integral inequality resulting from Gronwall (1919), i.e. if we substitute $a(t) = 0$ and $x(t)\, \dot{x}(t) = u(t)$ in (1), then from (4) and (9) we see that Theorem 1 reduces to the Gronwall's (1919) inequality. In the special cases when (i) $b(t) = 0$ and (ii) $a(t) = 0$, our Theorem 1 reduces to different inequalities which are new to the literature.

Another interesting and useful inequality is embodied in the following theorem.

Theorem 2 — Let $x(t)$, $\dot{x}(t)$, $a(t)$ and $b(t)$ be real-valued nonnegative continuous functions defined on I, and $x(0) = 0$, for which the inequality

$$x(t)\, \dot{x}(t) \leqslant k + M \left[x^2(t) + \int_0^t a(s)\, x(s)\, (x(s) + \dot{x}(s))\, ds \right.$$
$$\left. + \int_0^t b(s)\, x(s)\, \dot{x}(s)\, ds \right] \qquad \ldots(10)$$

holds for all $t \in I$, where k and M are nonnegative constants. Then

$$x(t) \leqslant \left[2k \int_0^t \left\{ \exp\left(\int_0^s M(2 + b(\tau))\, d\tau \right) \right.\right.$$
$$+ \int_0^s 2Ma(\tau) \exp\left(\int_\tau^s M(2 + b(\xi))\, d\xi \right)$$
$$\left.\left. \times \exp\left(\int_0^\tau (1 + M[2 + 2a(\xi) + b(\xi)])\, d\xi \right) d\tau \right\} ds \right]^{1/2} \qquad \ldots(11)$$

for all $t \in I$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 1. We omit the details.

We next establish the following integral inequality which can be used in some applications.

Theorem 3 — Let $x(t)\, \dot{x}(t)$, $a(t)$ and $b(t)$ be real-valued nonnegative continuous functions defined on I, and $x(0) = 0$; $W(r)$ be a positive, continuous, strictly increasing function for $r \geqslant 0$, and suppose further that the inequality

$$x(t)\,\dot{x}(t) \leqslant k + \int_0^t a(s)\,x(s)\,(x(s) + \dot{x}(s))\,ds + \int_0^t b(s)\,W(x(s)\,\dot{x}(s))\,ds \qquad ...(12)$$

is satisfied for all $t \in I$, where k is a nonnegative constant. Then for $0 \leqslant t \leqslant t_1$

$$x(t) \leqslant \Big[2 \int_0^t \Omega^{-1}\Big[\Omega\Big(k\Big\{1 + \int_0^\infty 2a(s) \exp\big(\int_0^s [1 + 2a(\tau)]\,d\tau\big)\,ds\Big\}\Big)$$

$$+ \int_0^s \Big\{b(\tau) + 2a(\tau) \int_0^\tau b(\xi) \exp\big(\int_\xi^\tau [1 + 2a(\rho)]\,d\rho\big)\,d\xi\Big\}\,d\tau\Big]\,ds\Big]^{1/2} \qquad ...(13)$$

where

$$\Omega(r) = \int_{r_0}^r \frac{ds}{W(s)}, \quad r \geqslant r_0 > 0 \qquad ...(14)$$

and Ω^{-1} is the inverse function of Ω, and $t_1 \in I$ is chosen so that

$$\Omega\Big(k\Big\{1 + \int_0^\infty 2a(s) \exp\big(\int_0^s [1 + 2a(\tau)]\,d\tau\big)\,ds\Big\}\Big)$$

$$+ \int_0^t \Big\{b(\tau) + 2a(\tau) \int_0^\tau b(\xi) \exp\big(\int_\xi^\tau [1 + 2a(\rho)]\,d\rho\big)\,d\xi\Big\}\,d\tau \in \text{Dom}\,(\Omega^{-1})$$

for all $t \in I$ such that $0 \leqslant t \leqslant t_1$.

PROOF: Define

$$m(t) = k + \int_0^t a(s)\,x(s)\,(x(s) + \dot{x}(s))\,ds$$

$$+ \int_0^t b(s)\,W(x(s)\,\dot{x}(s))\,ds, \; m(0) = k \qquad ...(15)$$

then by following the same steps as in the first part of the proof of Theorem 1 we have

$$\dot{m}(t) \leqslant 2a(t)\Big[m(t) + \int_0^t m(s)\,ds\Big] + b(t)\,W(m(t)). \qquad ...(16)$$

If we put

$$r(t) = m(t) + \int_0^t m(s)\,ds, \quad r(0) = m(0) = k \qquad ...(17)$$

then it follows by differentiating (17) and using the facts that

$$\dot{m}(t) \leqslant 2a(t)\, r(t) + b(t)\, W(m(t))$$

from (16) and $m(t) \leqslant r(t)$ from (17) we see that the inequality

$$\dot{r}(t) \leqslant [1 + 2a(t)]\, r(t) + b(t)\, W(m(t))$$

is satisfied which implies the estimation for $r(t)$ such that

$$r(t) \leqslant k \exp\left(\int_0^t [1 + 2a(s)]\, ds\right) + \int_0^t b(s)\, W(m(s)) \times \exp\left(\int_s^t [1 + 2a(\tau)]\, d\tau\right) ds.$$

Substituting this bound on $r(t)$ in (16) and using the monotone character of W we have

$$\dot{m}(t) \leqslant 2ka(t) \exp\left(\int_0^t [1 + 2a(s)]\, ds\right) + W(m(t)) \left[b(t) + 2a(t) \int_0^t b(s) \exp\left(\int_s^t [1 + 2a(\tau)]\, d\tau\right) ds \right]$$

which implies

$$m(t) \leqslant k + \int_0^\infty 2ka(s) \exp\left(\int_0^s [1 + 2a(\tau)]\, d\tau\right) + \int_0^t W(m(s)) \left[b(s) + 2a(s) \int_0^s b(\tau) \exp\left(\int_\tau^s [1 + 2a(\xi)]\, d\xi\right) d\tau \right] ds. \quad \ldots(18)$$

Define $v(t)$ by the right member of (18), then

$$\dot{v}(t) = W(m(t)) \left[b(t) + 2a(t) \int_0^t b(\tau) \exp\left(\int_\tau^t [1 + 2a(\xi)]\, d\xi\right) d\tau \right],$$

$$v(0) = k \left\{ 1 + \int_0^\infty 2a(s) \exp\left(\int_0^s [1 + 2a(\tau)]\, d\tau\right) ds \right\} \quad \ldots(19)$$

which in view of (18) implies

$$\dot{v}(t) \leqslant W(v(t)) \left[b(t) + 2a(t) \int_0^t b(\tau) \exp\left(\int_\tau^t [1 + 2a(\xi)]\, d\xi\right) d\tau \right] \quad \ldots(20)$$

Dividing both sides of (20) by $W(v(t))$, using (14) and integrating from 0 to t we obtain

$$\Omega(v(t)) - \Omega(v(0)) \leqslant \int_0^t \Big[b(s) + 2a(s) \int_0^s b(\tau) \times \exp\Big(\int_\tau^s [1 + 2a(\xi)]\, d\xi \Big)\, d\tau \Big] ds. \qquad \ldots(21)$$

Then from (21), (18) and the definition of $m(t)$ we have

$$x(t)\,\dot{x}(t) \leqslant \Omega^{-1}\Big[\Omega\Big(k \Big\{ 1 + \int_0^\infty 2a(s) \exp\Big(\int_0^s [1 + 2a(\tau)]\, d\tau \Big)\, ds \Big\} \Big) + \int_0^t \Big[b(s) + 2a(s) \int_0^s b(\tau) \exp\Big(\int_\tau^s [1 + 2a(\xi)]\, d\xi \Big)\, d\tau \Big] ds \Big]. \qquad \ldots(22)$$

Now, integrating both sides of (22) from 0 to t we obtain the desired bound in (13).

It is interesting to note that, if we substitute $a(t) = 0$ and $x(t)\,\dot{x}(t) = u(t)$ in (12), then from (22) we see that Theorem 3 reduces to the well-known integral inequality resulting from Bihari (1956).

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