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A DISTRIBUTIONAL GENERALIZED STIELTJES TRANSFORMATION

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A well-known generalization of Stieltjes transform is

$$F(x) = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \frac{\Gamma(\beta + 1)}{x}$$

$$\int_0^\infty \left(\frac{T}{x}\right)^\beta \cdot {}_2F_1\left(\beta + \eta + 1, \beta + 1; \alpha + \beta + \eta + 1; -\frac{T}{x}\right) f(T)\, dT$$

where $\beta \geqslant 0, \eta > 0$.

In this paper the above transform is extended to generalized functions (distributions). For this a testing function space over $(0, \infty)$ which contains the kernel of the above transform, is defined. The complex inversion formula for the above transform is also proved in distributional sense.

1. INTRODUCTION

Some generalization of the classical Stieltjes transform were given by many authors from time to time. Recently Joshi (1977) studied generalization of Stieltjes transform defined by

$$F(x) = \frac{\Gamma(\beta + \eta + 1)\,\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \int_0^\infty \frac{y^\beta}{x^{\beta+1}}\, {}_2F_1\left(\beta + \eta + 1, \beta + 1; \alpha + \beta + \eta + 1; -\frac{y}{x}\right) f(y)\, dy. \qquad ...(1.1)$$

He proved in the conventional sense the following complex inversion formula under certain conditions for the above generalization (1.1).

If $F(x)$ is given by (1.1), then

$$f(y) = \frac{1}{2\pi i} \lim_{w \to \infty} \int_{\sigma - iw}^{\sigma + iw} \frac{\Gamma(b + s - \beta - 1)}{\Gamma(a + s - \beta - 1)\,\Gamma(\beta + 1 - s)\,\Gamma(s)} \cdot y^{-s}\, M(s)\, ds \qquad ...(1.2)$$

where

$$M(s) = \int_0^\infty x^{s-1} F(x)\, dx \qquad ...(1.3)$$

$$a = \beta + \eta + 1,\ b = a + \alpha,\ s = \sigma + iw,\ \beta \geqslant 0,\ \eta > 0.$$

Several integral transforms have been extended to generalized functions by the author recently (see Tiwari 1979a, b, c). In this paper we extend the transform (1.1) and its complex inversion formula to generalized functions.

2. The Spaces of Type $J_{c,d}$ and Their Duals

In the following I is an open set $0 < T < \infty$. $D(I)$ is the space of all smooth functions on I having compact support. Let $\psi(T) \in D(I)$. Consider

$$L = \sup \{T : T \in \text{supp. } \psi\}$$

$$C_{k1} = \sup_{0 \leqslant T \leqslant L} \left| \mu_{c,d}(T) \sum_{p=0}^{k} C_p T^{1/2} D_T^p \psi \right|$$

where C_p's are constants and

$$\mu_{c,d}(T) = \begin{cases} T^c, & 1 \leqslant T < \infty \\ T^d, & 0 < T < 1. \end{cases}$$

We have for $0 < T < 1$, $k = 0, 1, ...$

$$\left| \mu_{c,d}(T)\, (TD_T)^k \{\sqrt{T}\, \psi(T)\} \right|$$

$$= \left| \mu_{c,d}(T) \sum_p C_p T^{(2p+1)/2} D_T^p\, \psi \right|$$

$$\leqslant \left| \mu_{c,d}(T) \sum_p C_p T^{1/2} D_T^p\, \psi \right|$$

$$\leqslant C_{k1} \qquad ...(2.1)$$

Also for $1 \leqslant T < \infty$

$$\left| \mu_{c,d}(T) \sum_p C_p T^{(2p+1)/2} D_t^p\, \psi \right|$$

$$\leqslant L^k \cdot C_{k1}. \qquad ...(2.2)$$

From (2.1) and (2.2) we have for $0 < T < \infty$,

$$\left| \mu_{c,d}(T)\, (TD_T)^k \{\sqrt{T}\, \psi\, (T)\} \right|$$

$$\leqslant C_{k1} + L^k C_{k1}$$

(equation continued on p. 1047)

$$\leqslant C_{k1}(1 + L^k)$$

$$\leqslant C_{k1}L_1^k \text{ where } L_1^k = 1 + L^k$$

$$\leqslant C_{k1}\left(\frac{L_1}{Ak^\alpha}\right)^k A^k k^{k\alpha} \qquad \ldots(2.3)$$

where $\alpha \geqslant 0$. From (2.3) it is clear that $\left(\frac{L_1}{Ak^\alpha}\right) \leqslant 1$ if and only if $k \geqslant \left(\frac{L_1}{A}\right)^{1/\alpha}$.

Define

$$k_0 = \left[\left(\frac{L_1}{A}\right)^{1/\alpha}\right] + 1$$

where $[x]$ denotes the Gaussian symbol that is the greatest integer not exceeding x.

Setting

$$C_{k2} \triangleq \max\left\{\frac{L_1}{A}, \left(\frac{L_1}{A_2^\alpha}\right)^2, \ldots, \left(\frac{L_1}{Ak_0^\alpha}\right)^{k_0}\right\} \qquad \ldots(2.4)$$

We get from (2.3)

$$\begin{aligned} &| \mu_{c,d}(T)\,(TD_T)^k \{\sqrt{T}\,\psi(T)\} | \\ &\leqslant C_{k1}\, C_{k2}\, A^k k^{k\alpha} \\ &\leqslant C_k\, A^k k^{k\alpha} \end{aligned} \qquad \ldots(2.5)$$

where $C_k = C_{k1} \cdot C_{k2}$

We now define the space $J_{c,d,\alpha}$.

A function $\psi(T)$ defined on $0 < T < \infty$ is said to be a member of $J_{c,d,\alpha}$ if $\psi(T)$ is smooth for $k = 0, 1, \ldots$ and

$$\begin{aligned} i_{c,d,k}(\psi) &= \sup_{0<t<\infty} | \mu_{c,d}(T)\,(TD_T)^k (\sqrt{T}\,\psi(T)) | \\ &\leqslant C_k A^k k^{k\alpha} \end{aligned} \qquad \ldots(2.6)$$

where the constants A and C_k depend on the testing function ψ. For $k = 0$, we set $K^{k\alpha} = 1$. The topology of the space $J_{c,d,\alpha}$ is that generated by the countable multinorms :

$\{i_{c,d,k}\}_{k=0}^{\infty}$. With this topology $J_{c,d,\alpha}$ is a countable multinormed space. A sequence $\{\psi_v\}_{v=1}$ is said to converge in $J_{c,d,\alpha}$ to ψ if for $k = 0, 1, \ldots$

$$i_{c,d,k}(\psi_v - \psi) \to 0 \text{ as } v \to \infty.$$

The following theorem can be easily proved.

Theorem 2.1 — $J_{c,d,\alpha}$ is complete and therefore a Frechet space.

We define the countable union space $J_{c,d}$ by

$$J_{c,d} = \bigcup_{\alpha_i=1}^{\infty} J_{c,d,\alpha_i}$$

Thus $J_{c,d}$ is the space of all testing functions (smooth) $\psi(T)$ on $0 < T < \infty$ such that

$$i_{c,d,k}\,\psi = \sup_{0<T<\infty} | \mu_{c,d}(T)\,(TD_T)^k \{\sqrt{T}\,\psi(T)\} |$$

$$< \infty. \qquad \ldots(2.7)$$

Since for each i, J_{c,d,α_i} is complete the countable union space $J_{c,d}$ is also complete.

From the discussion given in the introduction it is clear that $D(I) \subset J_{c,d,\alpha}$. The spaces $J'_{c,d,\alpha}$ and $J'_{c,d}$ denote the dual of the spaces $J_{c,d,\alpha}$ and $J_{c,d}$ respectively.

The space $J_{c,d}$ can also be defined in the following way :

Let $L_{c,d}$ denote the space of all complex valued smooth functions $\phi(t)$ on $-\infty < t < \infty$ on which functionals defined by

$$\gamma_k\phi = \gamma_{c,d,k}\phi(t)$$

$$= \sup_{-\infty<t<\infty} | \lambda_{c,d}(t)\, D_t^k\, \phi(t) | \qquad \ldots(2.8)$$

where $\lambda_{c,d}(t) = \begin{cases} e^{ct}, & 0 \leqslant t < \infty \\ e^{dt}, & -\infty < t < 0 \end{cases}$

assume finite values (see Zemanian 1968 p. 48).

Apply the change of variable $T = e^t$ to the definition of $L_{c,d}$ and set $\sqrt{T}\,\psi(T) = \phi(\log T)$ in (2.8). This yields the following definition.

Given two real numbers c and d, $J_{c,d}$ is the space of all smooth functions $\psi(T)$ on $0 < T < \infty$ such that

$$i_k\psi(T) = \sup_{0<T<\infty} | \mu_{c,d}(T)\,(TD_T)^k \{\sqrt{T}\,\psi(T)\} |$$

$$< \infty,\ k = 0, 1, 2 \ldots .$$

The following theorem can be easily proved.

Theorem 2.2 — The mapping

$$\psi(T) \mapsto \phi(t)$$

where $\psi(T) = T^{-1/2}\,\phi\,\log(T)$, $t = \log\, T$ is an isomorphism from $D(I)$ onto D. I is the interval $0 < T < \infty$. The inverse mapping is given by

$$\phi(t) \longmapsto \psi(T)$$

where $\qquad \phi(t) = e^{t/2}\, \psi(e^t).$

Let $L'_{c,d}$ be the dual space of the space $L_{c,d}$. If $f(t) \in L'_{c,d}$, we define $T^{-1/2} f(\log T)$ as a functional on $J_{c,d}$ by

$$\langle T^{-1/2} f(\log T),\, T^{-1/2}\, \phi(\log T)\rangle \triangleq \langle f(t), \phi(t)\rangle \qquad \ldots(2.9)$$

$$\phi(t) \in L_{c,d}.$$

It can be easily proved that the mapping defined by (2.9) is an isomorphism from D' onto $D'(I)$.

The mapping $f(t) \longmapsto T^{-1/2} f(\log T)$ as defined by (2.9) is an isomorphism from $L'_{c,d}$ onto $J'_{c,d}$.

3. The Distributional Generalized Stieltjes Transform

Theorem 3 1 — $\frac{\Gamma a}{\Gamma b}\, \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right)$ is in $J_{c,d}$ for $x > 0$, $c < \frac{1}{2}$ and $d > -\beta - \frac{1}{2}$.

Proof : For $\frac{\Gamma a}{\Gamma b}\, \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right)$ to be in $J_{c,d}$ we have to show that

$$\sup_{0<T<\infty} \left| \mu_{c,d}(T)\, (TD_T)^k \{\sqrt{T}\, \psi(T)\} \right|$$

is bounded where

$$\psi(T) = \frac{\Gamma a}{\Gamma b}\, \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right)$$

Consider

$$\sup_{0<T<\infty} \left| \mu_{c,d}(T)\, (TD_T)^k \left\{T^{1/2}\, Q T^{\beta}\, {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right)\right\}\right| \qquad \ldots(3.1)$$

(Q is a quantity independent of T).

Using the result

$$\frac{d}{dz}\, z^a\, {}_2F_1\,(a, b; c; z) = a z^{a-1}\, {}_2F_1\,(a+1; b; cz),$$

$$(3.1) = \sup_{0<T<\infty} \left| \mu_{c,d}(T)\, Q_1\, T^{\beta+1/2}\, {}_2F_1\left(a+k, \beta+1, b, -\frac{T}{x}\right)\right|$$

(Q_1 is quantity independent of T).

Further from Mathai and Saxena (1973, p. 307)

$$_2F_1\,(a, b; c, -x) = O(1) \text{ as } x \to 0.$$

Hence for $0 < T < 1$, $T \to 0$

$$(3.1) \leqslant \sup_{0<T<1} \mid T^d Q_1 T^{\beta+1/2}\, M_1 \mid$$

(M_1 is some constant)

That is as $T \to 0$, (3.1) is bounded

$$\text{if } d > -\beta - \tfrac{1}{2}.$$

Using the result (Mathai and Saxena 1973, p. 307)

$$2Fl\,(a, b; c; -x) = O(Ax^{-a} + Bx^{-b}),\ x \to \infty$$

where A and B are constants, we have as $T \to \infty$

$$(3.1) \leqslant \sup_{1<T<\infty} \left| T^c Q_2 T^{\beta+1/2}\, M_2 \left[A\left(\frac{T}{x}\right)^{-a} + B\left(\frac{T}{x}\right)^{-\beta-1} \right] \right|$$

(where M_2 is some constant).

Thus as $T \to \infty$, (3.1) is bounded if $c < \frac{1}{2}$.

Hence our theorem is proved.

We now define the distributional generalized Stieltjes transform.

If $f(T) \in J'_{c,d}$ for $c < \frac{1}{2}$ and $d > -\beta - \frac{1}{2}$, then the distributional generalized stieltjes transform $F(x)$ of $f(T)$ is defined by

$$F(x) = \left\langle f(T), \frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right) \right\rangle \quad \ldots(3.2)$$

where for each fixed $x(0 < x < \infty)$ the R.H.S. of (3.2) has a sense as the application of $f(T) \in J'_{c,d}$ to

$$\frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right) \in J_{c,d}.$$

Theorem 3.2 — Let $F(x)$ be defined by (3.2) in the distributional sense, then

$$F^{p}(x) = \left\langle f(T), \frac{\partial^p}{\partial x^p} \left[\frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{T}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{T}{x}\right) \right] \right\rangle. \quad \ldots(3.3)$$

Proof of the theorem is easy and hence omitted.

4. Complex Inversion Theorem

Lemma 4.1 — If $f \in J'_{c,d}$ and

$$\theta(x, u) = \frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta + 1)}{x} \left(\frac{u}{x}\right)^{\beta} {}_2F_1\left(a, \beta + 1; b; -\frac{u}{x}\right)$$

then

$$\int_0^\infty x^{s-1} \langle f(u), \theta(x, u) \rangle \, dx$$

$$= \langle f(u), \int_0^\infty \theta x^{s-1} dx \rangle \qquad \ldots(4.1)$$

Lemma 4.2 — Let $\psi \in D(I)$ and r be a fixed positive real number. If

$$P(s) = \int_0^\infty \psi(y)\, y^{-s} dy$$

where $s = \sigma + iw$ and $f \in J'_{c,d}$ then

$$\int_{-r}^{r} \langle f(u), u^{s-1} \rangle \, P(s)\, dw$$

$$= \langle f(u), \int_{-r}^{r} u^{s-1} P(s)\, dw \rangle \qquad \ldots(4.2)$$

Lemma 4.3 — If

(i) $\psi \in D(I)$.

(ii) σ and r are real numbers such that $\sigma > 1$, then

$$\frac{1}{\pi} \int_0^\infty \frac{\psi(y)}{u \log (u/y)} \left(\frac{u}{y}\right)^{\sigma} \sin\left(r \log \frac{u}{y}\right) dy$$

converges in $J_{c,d}$ to $\psi(u)$ as $r \to \infty$.

The proofs of Lemmas 4.1, 4.2 and 4.3 are similar to the proofs of similar results proved in Zemanian (1968, pp. 64-68; 1965, p. 121).

Theorem 4.1 — Let (i) $f \in J'_{c,d}$ and (ii) $\psi \in D(I)$, then

$$\lim_{r \to \infty} \left\langle \frac{l}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} \frac{\Gamma(b + s - \beta - 1)\, M(s)}{\Gamma(a + s - \beta - 1)\, \Gamma(\beta + 1 - s)\, \Gamma s} y^{-s} ds, \psi(y) \right\rangle$$

$$= \langle f, \psi \rangle \qquad \ldots(4.3)$$

where $M(s) = \int_0^\infty x^{s-1} F(x)\, dx.$

PROOF : We have

$$\left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \frac{\Gamma(b+s-\beta-1)\, M(s)}{\Gamma(a+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} y^{-s} ds,\ \psi(y) \right\rangle \quad \ldots(4.4)$$

$$= \int_0^\infty \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \frac{\Gamma(b+s-\beta-1)}{\Gamma(a+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} M(s)\, y^{-s}\, ds\, \psi(y)\, dy. \quad \ldots(4.5)$$

Denoting $\frac{\Gamma(b+s-\beta-1)}{\Gamma(a+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s}$ by Q, $s = \sigma + iw$ and changing the order of integration which is permissible here,

$$(4.5) = \frac{l}{2\pi} \int_{-r}^{r} QM(s) \int_0^\infty y^{-s}\psi(y)\, dy\, dw$$

$$= \frac{1}{2\pi} \int_{-r}^{r} Q \int_0^\infty x^{s-l} \left\langle f(u), \frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{u}{x}\right)^{\beta}. \right.$$

$$\left. \times\ 2Fl\left(a, \beta+1; b; -\frac{u}{x}\right) dx \right\rangle \int_0^\infty y^{-s}\psi(y)\, dy\, dw \quad \ldots(4.6)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} Q \left\langle f(u), \int_0^\infty x^{s-l} \frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{u}{x}\right)^{\beta}. \right.$$

$$\left. \times\ 2Fl\left(a, \beta+1, b; -\frac{u}{x}\right) dx \right\rangle \int_0^\infty y^{-s}\psi(y)\, dy\, dw. \quad \ldots(4.7)$$

(4.7) is obtained from (4.6) by using Lemma 4.1. By using the following result proved by Joshi (1977, pp. 17–18) :

$$Q. \int_0^\infty x^{s-l} \int_0^\infty \frac{\Gamma a}{\Gamma b} \frac{\Gamma(\beta+1)}{x} \left(\frac{u}{x}\right)^{\beta} {}_2F_1\left(a, \beta+1; b; -\frac{u}{x}\right). f(u)\, du\, dx$$

$$= \int_0^\infty u^{s-1} f(u)\, du.$$

We have

$$(4.7) \quad = \frac{l}{2\pi} \int_{-r}^{r} \langle f(u), u^{s-1} \rangle \int_{0}^{\infty} y^{-s}\psi(y)\, dy\, dw$$

$$= \left\langle f(u), \frac{1}{2\pi} \int_{-r}^{r} u^{s-1} \int_{0}^{\infty} y^{-s}\psi(y)\, dy\, dw \right\rangle \qquad \ldots(4.8)$$

(by using Lemma 4.2)

changing the order of integration in (4.8) we have

$$\left\langle f(u), \frac{l}{2\pi} \int_{0}^{\infty} \psi(y) \int_{-r}^{r} u^{s-1} y^{-s}\, dy\, dw \right\rangle. \qquad \ldots(4.9)$$

Now

$$\int_{-r}^{r} u^{s-1} y^{-s}\, dw$$

$$= \int_{-r}^{r} \left(\frac{u}{y}\right)^{\sigma} u^{-1} \left(\frac{u}{y}\right)^{iw} dw$$

$$= \int_{-r}^{r} \left(\frac{u}{y}\right)^{\sigma} (u)^{-1}\, e^{iw \log (u/y)}\, dw$$

$$= 2 \left(\frac{u}{y}\right)^{\sigma} u^{-1} \left[\log\left(\frac{u}{y}\right)\right]^{-1} \sin\left(r \log \frac{u}{y}\right)$$

Thus

$$(4.9) \quad = \left\langle f(u), \frac{1}{\pi} \int_{0}^{\infty} \psi(y) \left(\frac{u}{y}\right)^{\sigma} \sin\left(r \log \frac{u}{y}\right) \times \left[u \log\left(\frac{u}{y}\right)\right]^{-1} dy \right\rangle. \qquad \ldots(4.10)$$

By Lemma 4.3, $4.10 \to \langle f(u), \psi(u) \rangle$ as $r \to \infty$ which completes the proof of our theorem.

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