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# RELATIVISTIC ELECTROMAGNETIC FLUIDS AND TIME-LIKE FCRC MAPPINGS—I

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In this paper we have investigated certain symmetry mappings belonging to the family of contracted Ricci collineations (FCRC) admitted by thermally conducting electromagnetic fluid space-times for the symmetry vector collinear to the fluid flow vector. A number of theorems of physical interest have been established in the domain of local conservation laws involving FCRC quasi-symmetry properties admitted by thermally conducting electromagnetic fluid space-times.

## 1. INTRODUCTION

The relevance of relativistic magnetohydrodynamic (RMHD) effects in astrophysical situations has been emphasized by several workers (Lichnerowicz 1967, Yodzis 1971, Banerjee 1974, Date 1976 and Bekenstein and Oron 1978). Since RMHD flows are governed by a large number of coupled equations (Maxwell's, Euler's, conservation of energy and of baryon number), the solution of RMHD problem is very complicated. Looking at the recent trend of development in the study of RMHD flows, we find two avenues along which attempts can be made in making the solution process easier: (i) the search for geometrical device (Prasad 1979a, b, c, d); (ii) the study of conservation laws (Bekenstein and Oron 1978). In this paper we confine our attention on latter approach.

The recent study on conservation laws (Oliver and Davis 1976, Davis *et al.* 1976, Asgekar and Date 1977), however, has been concerned with the symmetry methods developed by Davis (1974). Norris *et al.* (1977) have developed a family of symmetry mappings which is called the family of contracted Ricci collineations (FCRC). The FCRC mappings form a symmetry chain which includes Ricci collineations (RC), affine collineations (AC), and motions (M). The FCRC symmetry mappings in RMHD has been studied in several papers (Prasad 1978a and Prasad and Sinha 1978, 1979).

The purpose of this paper is to study the FCRC symmetry mappings admitted by thermally conducting electromagnetic fluid space-times. In particular, we shall establish a set of theorems involving the FCRC quasi-symmetry mappings admitted by thermally conducting electromagnetic fluid space-times for the symmetry vector collinear to the fluid flow vector. These theorems throw light on the conservation expressions and Kinematical conditions corresponding to the members of the skeleton structures of time-like FCRC quasi-symmetry mappings admitted by such fluid space-times.

## 2. Kinematical Parameters and Field Equations

Let $x^i (i = 0, 1, 2, 3)$ be an arbitrary coordinate system in a region of space-time of signature $(+, -, -, -)$ and the metric tensor $g_{ij}$. The covariant derivative of the unit 4-velocity vector $u^i$ tangential to the world line is decomposed (Ehlers 1962)

$$u_{i;j} = \sigma_{ij} + \omega_{ij} + \theta\gamma_{ij} + Du_i u_j \qquad \text{...(2.1)}$$

where the scalar $\theta = \frac{1}{3} u^i_{;i}$ is the expansion, $Du^i = u^i_{;j} u^j$ is the fluid acceleration vector, $\sigma_{ij} = u_{(i;j)} - Du_{(i} u_{j)} - \theta\gamma_{ij}$ is the shear tensor, $\omega_{ij} = u_{[i;j]} - Du_{[i} u_{j]}$ is the rotation tensor, $\gamma_{ij} = g_{ij} - u_i u_j$ is the projection tensor. Here the round and square brackets around the indices denote symmetrization and skew-symmetrization respectively and the semicolon stands for covariant differentiation. $D$ represents the directional derivative along the stream line.

The field equations governing the behaviour of relativistic electromagnetic fluids are (Lichnerowicz 1967)

$$R_{ij} - \frac{1}{2} g_{ij} R = T_{ij} \qquad \text{...(2.2)}$$

where the stress-energy momentum tensor $T_{ij}$ for thermally conducting electromagnetic fluid is given by

$$T_{ij} = (\overset{*}{\rho} + \overset{*}{p}) u_i u_j - \overset{*}{p} g_{ij} - (\lambda e_i e_j + \mu h_i h_j) + P_i u_j + P_j u_i \qquad \text{...(2.3)}$$

where

$$\overset{*}{\rho} = \rho + \frac{1}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \text{...(2.4)}$$

$$\overset{*}{p} = p + \frac{1}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \text{...(2.5)}$$

and

$$P_i = q_i - V_i. \qquad \text{...(2.6)}$$

Here $\rho$ is the matter energy density of the fluid, $p$ the isotropic pressure, $\overset{*}{\rho}$ the total energy density of thermally conducting electromagnetic fluid, $\overset{*}{p}$ the total pressure, $\mid e \mid$ the magnitude of electric field vector $e^i$, $\mid h \mid$ the magnitude of magnetic field vector $h^i$, $\mu$ the magnetic permeability, $\lambda$ the electric permitivity, $q^i$ the heat flux vector, $V^i$ the electromagnetic energy flux vector and $P^i$ the 'energy flux' (Prasad 1978b) vector. The expression for heat energy flux vector (Eckart 1940) is given by

$$q^i = K(T_{,j} - TDu_j)\,\gamma^{ij} \qquad ...(2.7)$$

where $K$ is the coefficient of heat conduction and $T$ is the rest temperature. The relations (Lichnerowicz 1967) connecting the thermodynamical variables are

$$TDS = D_i + pD(1/\rho_0) \qquad ...(2.8)$$

$$S^i = \rho_0\, Su^i + q^i/T \qquad ...(2.9)$$

$$\rho = \rho_0(1 + i) \qquad ...(2.10)$$

$$\chi = 1 + i + p/\rho_0, \qquad ...(2.11)$$

where $\rho_0$ is the proper matter density, $i$ the internal energy density, $\chi$ the fluid index and $S^i$ the entropy-flux vector. The equations of stream lines are

$$\overset{*}{\rho}\, Du^i - \gamma^i_j\, S^{jk}_{;k} + \gamma^i_j\, DP^j + 3\theta P^i + P^j u^i_{;j} = 0 \qquad ...(2.12)$$

where

$$S^{jk} = \overset{*}{p}\, \gamma^{jk} + (\lambda e^j e^k + \mu h^j h^k) \qquad ...(2.13)$$

## 3. FCRC Symmetry Mappings

In this section we shall study the time-like FCRC symmetry properties admitted by the thermally conducting electromagnetic fluid space-times for the symmetry vector along the fluid flow vector. The FCRC symmetry properties are defined by the conditional relations (Norris *et al.* 1977).

$$\mathcal{L}_\xi R_{ij} = H_{ij},\; g^{ij}H_{ij} = 0 \qquad ...(3.1)$$

where $H_{ij}$ is any trace-free symmetric tensor which is not identically equal to $\mathcal{L}_\xi R_{ij}$. Here $\mathcal{L}_\xi$ denotes the Lie differentiation with respect to the time-like vector $\xi^i$ parallel to the fluid flow vector. When a vector $\xi^i$ satisfying (3.1) is determined to within a multiplicative constant for a particular $H_{ij}$ in a given space-time then we say that the given space-time admits this particular FCRC symmetry property. If this particular $H_{ij}$ partially determines $\xi^i$, then the symmetry property is called an FCRC quasi-symmetry property. It can be shown that the conservation expression (Davis *et al.* 1976)

$$(\sqrt{-g}\, R^i_j\, \xi^j)_{;i} = 0 \qquad ...(3.2)$$

holds for both FCRC symmetry and quasi-symmetry properties.

By virtue of (2.2) and (2.3), we have

$$R_{ij} = Au_i u_j - B\gamma_{ij} - \mu h_i h_j - \lambda e_i e_j + P_i u_j + P_j u_i \qquad ...(3.3)$$

where

$$2A = \rho + 3p + \mu \mid h \mid^2 + \lambda \mid e \mid^2 \qquad ...(3.4)$$

$$2B = \rho - p + \mu \mid h \mid^2 + \lambda \mid e \mid^2. \qquad ...(3.5)$$

To specify FCRC symmetry properties defined by (3.1), we decompose $\mathcal{L}_\xi R_{ij}$ with $\xi^i = \varphi u^i$ as follows:

$$\mathcal{L}_\xi R_{ij} = H_{ij} + \tfrac{1}{2} g_{ij} [(\overset{*}{\rho} + S)\xi^k + 2\varphi P^k]_{;k} \qquad ...(3.6)$$

where

$$\begin{aligned} H_{ij} = {} & \psi [u_i u_j - \tfrac{1}{4} g_{ij}] - \varphi[(\overset{*}{\rho} - \tfrac{1}{3}S)\sigma_{ij}] \\ & - \varphi[\mathcal{L}_u \overset{T}{S}_{ij} - \tfrac{1}{3}\gamma_{ij}(\gamma^{km}\mathcal{L}_u \overset{T}{S}_{km})] \\ & + (\overset{*}{\rho} + S)\, u_{(i}\, \gamma^k_{j)}\, [\varphi_{,k} + \varphi D u_k] \\ & + 2\varphi [DP_k \gamma^k_{(i} u_{j)}] + 2\theta\varphi [P_{(i} u_{j)}] \\ & + 2\varphi [P_k \sigma^k_{(i} u_{j)}] + 2\varphi [P_k \omega^k_{(i} u_{j)}] \\ & + \varphi[2P_{(i} Du_{j)} - \tfrac{1}{3}\gamma_{ij}(2P^k Du_k)] \\ & + [2P_{(i}\varphi_{,j)} - \tfrac{1}{3}\gamma_{ij}(2P^k\varphi_{,k})]. \end{aligned} \qquad ...(3.7)$$

Here

$$\psi = \tfrac{1}{3} [\{(\overset{*}{\rho} + S)\,\xi^k\}_{;k} + 2(\overset{*}{\rho} + S)(D\varphi - \theta\varphi) - 2(\varphi P^k)_{;k}] \qquad ...(3.8)$$

and

$$\overset{T}{S}_{ij} \overset{def}{=} S_{ij} - \tfrac{1}{3}\gamma_{ij} S. \qquad ...(3.9)$$

It is observed, in view of (3.2), that the last term in (3.6) vanishes and represents the conservation expression for FCRC symmetry vector $\xi^i$. Further, from (3.7), it is obvious that each individual bracketed term is trace-free. To discuss FCRC symmetry properties admitted by the thermally conducting electromagnetic fluid space-times, we focus attention on the skeleton structures formed by the FCRC quasi-symmetry properties because the role of the skeleton structures is important for the study of actual symmetry properties. The trace-free property of each individual term in (3.7) allows one to make different types of choices for the tensor $H_{ij}$ in the manner of Norris *et al.* (1977) as follows :

$$\mathcal{L}_\xi R_{ij} = \overset{1}{H}_{ij} = H_{ij} - \psi [u_i u_j - \tfrac{1}{4} g_{ij}] \qquad ...(3.10)$$

$$\mathcal{L}_\xi R_{ij} = \overset{2}{H}_{ij} = H_{ij} + \varphi[(\overset{*}{\rho} - \tfrac{1}{3} S)\, \sigma_{ij}] \qquad ...(3.11)$$

$$\mathcal{L}_\xi R_{ij} = \overset{3}{H}_{ij} = H_{ij} + \varphi[\mathcal{L}_u \overset{T}{S}_{ij} - \tfrac{1}{3}\gamma_{ij}(\gamma^{km}\mathcal{L}_u \overset{T}{S}_{km})] \qquad ...(3.12)$$

$$\mathcal{L}_\xi R_{ij} = \overset{4}{H}_{ij} = H_{ij} - (\overset{*}{\rho} + S) u_{(i}\gamma^k{}_{j)} [\varphi_{,k} + \varphi D u_k] \qquad ...(3.13)$$

$$\mathcal{L}_\xi R_{ij} = \overset{5}{H}_{ij} = H_{ij} - 2\theta\varphi \, [P_{(i} \, u_{j)}] \qquad ...(3.14)$$

$$\mathcal{L}_\xi R_{ij} = \overset{6}{H}_{ij} = H_{ij} - 2\varphi \, [P_k \sigma^k{}_{(i} u_{j)}] \qquad ...(3.15)$$

$$\mathcal{L}_\xi R_{ij} = \overset{7}{H}_{ij} = H_{ij} - 2\varphi \, [P_k \, \omega^k{}_{(i} u_{j)}]. \qquad ...(3.16)$$

Similarly, other types of choices can be made but we are interested in the analysis of the above mentioned skeleton structures which will be discussed one by one in the form of theorems.

*Theorem* 3.1 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.10) iff (i) $[(\overset{*}{\rho} + S)\,\xi^k + 2\varphi \, P^k]_{;k} = 0$ and (ii) $[(\overset{*}{\rho} + S)\,\varphi^2 V^{2/3}]$ is constant along the flow lines.

PROOF : First, we assume that the thermally conducting electromagnetic fluid space-time admits the symmetry property (3.10), i.e., $\mathcal{L}_\xi R_{ij} = \overset{1}{H}_{ij}$ for the symmetry vector $\xi^i = \varphi u^i$. Setting $\mathcal{L}_\xi R_{ij} = \overset{1}{H}_{ij}$ in the identity map (3.6), we get

$$\psi(u_i u_j - \tfrac{1}{4} g_{ij}) + \tfrac{1}{2} g_{ij} \, [(\overset{*}{\rho} + S)\,\xi^k + 2\varphi P^k]_{;k} = 0. \qquad ...(3.17)$$

Contracting (3.17) with $g^{ij}$, we find

$$[(\overset{*}{\rho} + S)\,\xi^k + 2\varphi P^k]_{;k} = 0. \qquad ...(3.18)$$

In view of (3.18) and (3.8), it follows from (3.17) that

$$[\{(\overset{*}{\rho} + S)\,\xi^k\}_{;k} + 2(\overset{*}{\rho} + S)\,(D\varphi - \theta\varphi) - 2\,(\varphi \, P^k)_{;k}] = 0 \qquad ...(3.19)$$

since $(u_i u_j - \tfrac{1}{4} g_{ij}) \neq 0$.

Again the use of (3.18) reduces (3.19) to

$$[\{(\overset{*}{\rho} + S)\,\xi^k\}_{;k} + (\overset{*}{\rho} + S)\,(D\varphi - \theta\varphi)] = 0. \qquad ...(3.20)$$

Using the definition (Ehlers 1961) $3\theta = \dfrac{1}{V} DV$, where $V$ is the volume element of the fluid, in eqn. (3.20) we obtain

$$D \, \log \, [(\overset{*}{\rho} + S)\,\varphi^2 V^{2/3}] = 0 \qquad ...(3.21)$$

which implies that $[(\overset{*}{\rho} + S)\,\varphi^2 V^{2/3}]$ is constant along the fluid flow lines. Further, assuming that the conditions (i) and (ii) hold and reversing the steps of the above proof, we arrive at $\mathcal{L}_\xi R_{ij} = \overset{1}{H}_{ij}$. This proves the converse of the theorem. We now interpret the conservation law (3.18) physically. It follows from (2.4), (2.5), (2.13) and (3.18) that

$$[(\rho + 3p + \lambda \mid e \mid^2 + \mu \mid h \mid^2)\, \xi^k + 2\varphi P^k]_{;k} = 0, \qquad ...(3.22)$$

which shows that only a combination of active gravitational density of thermally conducting electromagnetic fluid and energy flux is conserved.

*Theorem* 3.2 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.11) iff (i) $[(\overset{*}{\rho} + S)\, \xi^k + 2\varphi\, P^k]_{;k} = 0$ and (ii) either $\sigma_{ij} = 0$ or $p = \rho + \frac{1}{3}(\lambda \mid e \mid^2 + \mu \mid h \mid^2)$.

PROOF : First, we assume that the thermally conducting electromagnetic fluid space-time admits the symmetry property (3.11), i.e., $\mathcal{L}_\xi R_{ij} = \overset{2}{H}_{ij}$ for the symmetry vector $\xi^i = \varphi u^i$. Setting $\mathcal{L}_\xi R_{ij} = \overset{2}{H}_{ij}$ in the identity (3.6), we obtain

$$\varphi[(\overset{*}{\rho} - \tfrac{1}{3} S)\, \sigma_{ij}] + \tfrac{1}{2}\, g_{ij}[(\overset{*}{\rho} + S)\, \xi^k + 2\varphi P^k]_{;k} = 0. \qquad ...(3.23)$$

Contraction of (3.23) with $g^{ij}$ gives

$$[(\overset{*}{\rho} + S)\, \xi^k + 2\varphi P^k]_{;k} = 0 \qquad ...(3.24)$$

which establishes the condition (i).

On using (3.24) in (3.23), we have

$$(\overset{*}{\rho} - \tfrac{1}{3} S)\, \sigma_{ij} = 0 \qquad ...(3.25)$$

which implies that either $\sigma_{ij} = 0$ or $\overset{*}{\rho} = \frac{1}{3} S$. This proves the condition (ii).

*Theorem* 3.3 — If a space-time filled with thermally conducting electromagnetic fluid in "restricted steady state" in which the electric field and fluid vorticity are aligned admits an FCRC quasi-symmetry property (3.12), then the magnetic field vector is an eigen vector of fluid vorticity tensor.

PROOF : First, we assume that the thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.12), i.e., $\mathcal{L}_\xi R_{ij} = \overset{3}{H}_{ij}$ for the symmetry vector $\xi^i = \varphi u^i$. Setting $\mathcal{L}_\xi R_{ij} = \overset{3}{H}_{ij}$ in the identity map (3.6) and using (3.2), we get

$$\mathcal{L}_u \overset{T}{S}_{ij} - \tfrac{1}{3}\, \gamma_{ij}\, (\gamma^{km}\, \mathcal{L}_u \overset{T}{S}_{km}) = 0. \qquad ...(3.26)$$

Since the thermally conducting electromagnetic fluid is in 'restricted steady state' (Esposito and Glass 1977), it follows that

$$\gamma^i_k De^k = \gamma^i_k Dh^k = \theta = \sigma_{ik} h^k = 0. \qquad ...(3.27)$$

On account of (3.26) and (3.27), one may obtain

$$(\tfrac{1}{3} DW) \gamma_{ij} + \tfrac{2}{3} W\sigma_{ij} + \lambda [2\sigma_{k(i}e^k e_{j)} + 2\omega_{k(i}e^k e_{j)}] + 2\mu\omega_{k(i}h^k h_{j)}$$
$$- \tfrac{2}{3} \gamma_{ij} (\lambda\sigma_{km} e^k e^m) = 0 \qquad ...(3.28)$$

where $W = (\lambda \mid e \mid^2 + \mu \mid h \mid^2)$.

On using the condition that the electric field and fluid vorticity vectors are aligned, i.e., $e^k\omega_{ki} = 0$ in (3.28), we get

$$(\tfrac{1}{3} DW) \gamma_{ij} + \tfrac{2}{3} W\sigma_{ij} + 2\lambda\sigma_{k(i} e^k e_{j)} + 2\mu\omega_{k(i} h^k h_{j)}$$
$$- \tfrac{2}{3}\gamma_{ij} (\lambda\sigma_{km} e^k e^m) = 0. \qquad ...(3.29)$$

On contracting (3.29) with $h^i$ and making use of the condition $\sigma_{ik} h^k = 0$ of (3.27) and orthogonality of electric and magnetic fields, we obtain

$$\mu \mid h \mid^2 \omega_{kj} h^k + \tfrac{1}{3} [2\lambda\sigma_{km} e^k e^m - DW] h_j = 0 \qquad ...(3.30)$$

which shows that $h^k$ is an eigen vector of $\omega_{kj}$. This proves the statement. We would now state the following theorems without giving proofs because their proofs follow from the identity map (3.6) in a way similar to those of the proofs of the preceding theorems.

*Theorem* 3.4 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.13) iff (i) the fluid acceleration vector is curl-free and (ii) $[(\overset{*}{\rho} + S) \xi^k + 2\varphi P^k]_{;k} = 0$.

*Theorem* 3.5 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.14) iff (i) the congruence of fluid flow lines is non-expanding and (ii) $[(\overset{*}{\rho} + S)\xi^k + 2\varphi P^k]_{;k} = 0$.

*Theorem* 3.6 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.15) iff (i) the "energy flux" vector is an eigen-vector of fluid shear tensor with zero eigenvalue and (ii) $[(\overset{*}{\rho} + S) \xi^k + 2\varphi P^k]_{;k} = 0$.

*Theorem* 3.7 — A thermally conducting electromagnetic fluid space-time admits an FCRC quasi-symmetry property (3.16) iff (i) the "energy flux" vector and fluid vorticity vector are aligned and (ii) $[(\overset{*}{\rho} + S) \xi^k + 2\varphi P^k]_{;k} = 0$.

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