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STRONG MAGNETOGASDYNAMIC SPHERICAL SHOCKS WITH INCREASING ENERGY

B. G. VERMA AND J. B. SINGH

Department of Mathematics, University of Gorakhpur, Gorakhpur 273001

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Adiabatic and isothermal flows driven by an expanding piston are investigated with an approximate method suggested by Laumbach and Probstein (1969). A comparative study has been made between the flow variables for both the cases. The total energy of the flow between the shock and piston is taken to be dependent on the shock radius obeying a power law. The shock is assumed to be strong and propagating into a perfect gas at rest with non-uniform density in a non-uniform conducting medium.

1. INTRODUCTION

The problem considered in this paper is that of adiabatic and isothermal flows behind a strong shock produced by an expanding piston and propagating into a non uniform atmosphere at rest. The body forces due to earth's gravitational and wind effects are neglected and the effects of magnetic field have been taken into account. The atmosphere is considered to be initially at rest and cold, that is, at zero temperature and pressure.

Ahead of the shock, the density distribution is taken to vary as an inverse power of radial distance from the point explosion, i.e.

$$\rho_0 = Ar_0^{-w} \ (w > 0) \qquad \ldots(1.1)$$

where A, w are constants, r_0 is the initial coordinate of the fluid particle.

Rosenau and Frankenthal (1976) have taken a model which may be regarded as a first approximation to the distant flow of a stellar wind in a narrow conical slab centered on the ecliptic plane. There, the stretching of the dipolar field of a rotating star produces the appropriate field and flow. The radial component of the field behaves as r^{-2} and may be ignored by comparision with the azimuthal field which behaves as r^{-1}, where the flow is primarily radial. The choice of an azimuthal field allows only the possibility of fast shocks. In order to give a meaning to the otherwise physically unrealizable magnetic field with spherical symmetry, Summers (1975) has taken the magnetic field which is replaced by an idealized field such that the lines of force lie on hemispheres whose centre is the explosion. Hence ahead of the shock, the magnetic field distribution is assumed as

$$h_0 = Br_0^{-1} \qquad \ldots(1.2)$$

and directed tangential to the advancing shock front, where B is constant.

The total energy of the flow between the piston and shock obeys the power law,

$$E = ER^n \quad (n > 0) \qquad \ldots(1.3)$$

where C, n are constants and R is the shock radius.

Only spherical case is considered here by applying the technique developed by Laumbach and Probstein (1969).

The flow behind the shock is rendered approximately isothermal when the flows take place at very high temperature, the temperature gradient being zero throughout the flow.

In the last section a comparative study has been made between the adiabatic and isothermal flow variables.

2. Equations for Adiabatic Flow

We assume that most of the flow is concentrated in the layer, known as the shock layer, next to the shock. Another main point is the Taylor's expansion of the Eulerian coordinate r, a function of Lagrangian coordinate r_0 and time t, in r_0 with the expansion parameterised in t through the Taylor-coefficients and the shock radius r. The equations of continuity, momentum, magnetic field and energy in the integrated form, are

$$\int_{r_p}^{R} r^2 dr = \int_0^R \frac{\rho_0}{\rho} r_0^2 \, dr_0 \qquad \ldots(2.1)$$

where subscript p refers to condition when particle is at piston,

$$p^*(r_0, t) - p_s^*(R) = \int_{r_0}^{R} \frac{1}{r^2} \frac{\partial^2 r}{\partial t^2} \rho_0 r_0^2 \, dr_0 + \int_{r_0}^{R} \frac{h^2}{\rho r^3} \rho_0 r_0^2 \, dr_0 \qquad \ldots(2.2)$$

where $p^* = p + \frac{1}{2} h^2$ and the subscript s refers to condition when the particle is at the shock front,

$$h(r_0, t) - h_s(R) = \int_{r_0}^{R} \left[\frac{1}{r^2} + \frac{1}{ru^2} \frac{\partial^2 r}{\partial t^2} \right] h_0 r_0 \, dr_0 \qquad \ldots(2.3)$$

and

$$\frac{E}{4\pi} = \frac{CR^n}{4\pi} = \int_{r_p}^{R} \frac{p}{(\gamma - 1)} r^2 dr + \int_{r_p}^{R} \frac{h^2}{2} r^2 \, dr + \int_{0}^{R} \tfrac{1}{2} \left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^2 \, dr_0. \quad ...(2.4)$$

For adiabatic flow we can write

$$\frac{\rho(r_0, t)}{\rho_s(r_0)} = \left[\frac{p(r_0, t)}{p_s(r_0)}\right]^{1/\gamma}. \quad ...(2.5)$$

Since the magnetic field is 'frozen in' we have,

$$hr \, dr = h_0 r_0 \, dr_0. \quad ...(2.6)$$

Here the cylindrical form of the magnetic field equation is taken as in Summers (1975) and Rosenau and Frankenthal (1976).

The energy eqn. (2.4) with the help of eqn. (2.6) can be written as

$$\frac{E}{4\pi} = \frac{CR^n}{4\pi} = \int_{r_p}^{R} \frac{p}{\gamma - 1} r^2 \, dr + \tfrac{1}{2} \int_{0}^{R} \left(\frac{\partial r}{\partial r_0}\right)^{-1} h_0^2 r_0^2 \, dr_0 + \tfrac{1}{2} \int_{0}^{R} \left(\frac{\partial r}{\partial t}\right)^2 \rho_0 r_0^2 \, dr_0. \quad ...(2.7)$$

Similarly eqn. (2.2) with the help of eqn. (2.6) can be written as

$$p^*(r_0, t) - p_s^*(R) = \int_{r_0}^{R} \frac{1}{r^2} \left(\frac{\partial^2 r}{\partial t^2}\right) \rho_0 r_0^2 \, dr_0 + \int_{r_0}^{R} \frac{1}{r^3} \left(\frac{\partial r}{\partial r_0}\right)^{-1} h_0^2 r_0^2 \, dr_0. \quad ...(2.8)$$

With strong shock assumption we have

$$\rho_s = \frac{\gamma + 1}{\gamma - 1} \rho_0 \quad ...(2.9)$$

$$h_s = \frac{\gamma + 1}{\gamma - 1} h_0 \quad ...(2.10)$$

$$p_s = \frac{2}{\gamma + 1} \rho_0 \dot{R}^2 \quad ...(2.11)$$

and

$$u_s = \left(\frac{\partial r}{\partial t}\right)_s = \frac{2}{\gamma + 1} \dot{R} \quad ...(2.12)$$

where a dot above a symbol refers to differentiation with respect to time.

The integral in eqn. (2.8) is evaluated approximately by replacing the integrands $\frac{1}{r^2}\left(\frac{\partial^2 r}{\partial t^2}\right)$ and $\frac{1}{r^3}\left(\frac{\partial r}{\partial r_0}\right)^{-1}$ by their values at the shock (see appendix I). Then using (1.1), (1.2), (2.10), (2.11) and (2.3) we get

$$p(r_0, t) = \frac{2}{\gamma+1}\rho_0\dot{R}^2 + \tfrac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)^2 h_0^2 + \left[\frac{A}{R^2}\left(\frac{\partial^2 r}{\partial t^2}\right)_s\right] \times \frac{1}{3-w}(R^{3-w} - r_0^{3-w}) + \left[\frac{B^2}{R^3}\left(\frac{\gamma+1}{\gamma-1}\right)\right][R - r_0] - \tfrac{1}{2}\left[\frac{\gamma+1}{\gamma-1}h_0 + \left\{\frac{1}{R^2} + \frac{1}{R}\left(\frac{\partial r}{\partial t}\right)_s^{-2}\left(\frac{\partial^2 r}{\partial t^2}\right)_s\right\}B(R - r_0)\right]^2. \quad \text{...(2.13)}$$

Now we consider the integrals of eqn. (2.7). Replacing the integrand $p(r_0, t)$ by $p(0, t)$ which can be found out from (2.13), $\frac{\partial r}{\partial t}$ by $\left(\frac{\partial r}{\partial t}\right)_s$ and $\frac{\partial r}{\partial r_0}$ by $\left(\frac{\partial r}{\partial r_0}\right)_s$; eqn. (2.7), after integration, reduces to a second order differential equation in R, which after a couple of integration yields

$$R = Kt^{\delta} \quad \text{...(2.14)}$$

where $\delta = \frac{2}{5 - w - n}$ and K is a constant.

We have found that this constant of proportionality is insignificant while obtaining the solutions in the form u/u_s, ρ/ρ_s, p/p_s and h/h_s as a function of $\lambda(= r/R)$. Here $R = r_p/\lambda_p$, where λ_p is a value of λ at the piston.

Using (2.14), (App. I.10), (2.9), (2.10), (2.3) and (2.5) we get

$$\frac{p(r_0, t)}{p_s(R)} = F^{\gamma}(\xi)\, \xi^{-[\delta(w-2)+2]/\delta} \quad \text{...(2.15)}$$

$$\frac{\rho(r_0, t)}{\rho_s(R)} = F(\xi)\, \xi^{-w} \quad \text{...(2.16)}$$

$$\rho(r_0, t) = A\frac{\gamma+1}{\gamma-1}F(\xi)\,(R\xi)^{-w} \quad \text{...(2.17)}$$

and

$$\frac{h(r_0, t)}{h_s(R)} = 1 + H(1 - \xi) \quad \text{...(2.18)}$$

where $$\xi = \frac{r_0}{R} \quad \text{...(2.19)}$$

$$F(\xi) = \frac{\rho(r_0, t)}{\rho_s(r_0)} = \left[1 + G(1 - \xi^{3-w}) + \frac{(\gamma + 1)^2 (1 - \xi)}{2(\gamma - 1) M_h^2} + \frac{(\gamma + 1)^3}{4(\gamma - 1)^2 M_h^2} \{1 - \langle 1 + H(1 - \xi) \rangle^2\} \right]^{1/\gamma} \xi^{[\delta(w-2)+2]/\delta\gamma} \quad \ldots(2.20)$$

$$G = \frac{\gamma + 1}{2(3 - w)} \left[\frac{M_h^2}{M_h^2 + \left(\frac{\gamma + 1}{\gamma - 1}\right)^2} \right] \left[\left\{ \frac{4(2\gamma - 1)}{(\gamma + 1)^2} + \frac{2(\gamma + 1)}{(\gamma - 1)^2 M_h^2} \right\} \frac{\delta - 1}{\delta} - \left\{ \frac{2(\gamma - 1)}{(\gamma + 1)^3} \langle w(\gamma + 1) - 4\gamma \rangle \right\} \right]$$

and

$$H = \left(\frac{\gamma - 1}{\gamma + 1} \right) + \frac{(\gamma^2 - 1)}{4} \left[\frac{M_h^2}{M_h^2 + \left(\frac{\gamma + 1}{\gamma - 1}\right)^2} \right] \left[\left\{ \frac{4(2\gamma - 1)}{(\gamma + 1)^2} + \frac{2(\gamma + 1)}{(\gamma - 1)^2 M_h^2} \right\} \frac{\delta - 1}{\delta} - \left\{ \frac{2(\gamma - 1)}{(\gamma + 1)^3} \langle w(\gamma + 1) - 4\gamma \rangle \right\} \right]$$

where $M_h = \frac{\sqrt{\rho_0}\dot{R}}{h_0}$, is known as Alfvén Mach number.

From eqns. (2.1), (2.9), (2.13) and (2.19) we get

$$1 - \lambda_p^3 = 3 \frac{\gamma - 1}{\gamma + 1} \int_0^1 \frac{x^2}{F(x)} dx \quad \ldots(2.21)$$

$$\lambda^3 - \lambda_p^3 = 3 \frac{\gamma - 1}{\gamma + 1} \int_0^\xi \frac{x^2}{F(x)} dx \quad \ldots(2.22)$$

and

$$\frac{u(r_0, t)}{u_s(R)} = \frac{\gamma + 1}{2\lambda^2} \left[\lambda_p^3 + \frac{\gamma - 1}{\gamma + 1} \left\{ \frac{\xi}{\rho} \left(\frac{\partial \rho}{\partial \xi} \right)_{t=\text{const.}} - \frac{R}{\dot{R}} \frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} \right)_{\xi=\text{const.}} \right\} \xi^2 \, d\xi \right]. \quad \ldots(2.23)$$

Using (2.17), eqn. (2.23) can be simplified as

$$\frac{u}{u_s} = \frac{\gamma + 1}{2\lambda^2} \left[\lambda_p^3 + \frac{\gamma - 1}{\gamma + 1} \int_0^{\xi} \frac{1}{F(x)} \left\{ x \frac{F'(x)}{F(x)} + w \right\} x^2 \, dx. \right. \quad \ldots(2.24)$$

We use the identity (2.21) to calculate the value of λ_p while the eqn. (2.22) is used to find the relation between the reduced Eulerian coordinate and the Lagrangian coordinate. This relation helps to calculate the values of u/u_s, ρ/ρ_s, p/p_s and h/h_s as functions of λ.

3. Equations for Isothermal Flow

The Lagrangian equations of continuity, momentum, magnetic field being the same as in section 2, for isothermal flow the energy equation, we have

$$\frac{\partial T}{\partial r} = 0. \quad \ldots(3.1)$$

Using the eqn. (3.1) and the equation of state $p = \Gamma \rho T$, where Γ is a gas constant and T the temperature, we obtain

$$\frac{p}{p_s} = \frac{\rho}{\rho_s}. \quad \ldots(3.2)$$

The variable $\lambda (= r/R)$ assumes the value 1 at the shock front and λ_p on the piston surface. This enables us to express the piston radius $r_p = \lambda_p R$. The shock conditions are given by

$$\rho_s = \frac{1}{\beta} \rho_0 \quad \ldots(3.3)$$

$$h_s = \frac{1}{\beta} h_0 \quad \ldots(3.4)$$

$$p_s = (1 - \beta) \rho_0 \dot{R}^2 \quad \ldots(3.5)$$

and

$$u_s = (1 - \beta) \dot{R}. \quad \ldots(3.6)$$

Equation (2.8) also holds in this case and it is evaluated approximately by replacing $\partial r/\partial t$, $\partial r/\partial r_0$ and $\partial^2 r/\partial t^2$ by their values at the shock front which are given in (App. II). The eqn. (2.13) for the isothermal case is given as below

$$p(r_0, t) = (1 - \beta) \rho_0 \dot{R}^2 + \frac{h_0^2}{2\beta^2} + \left[\frac{A}{R^2} \left(\frac{\partial^2 r}{\partial t^2} \right)_s \right] \frac{1}{(3 - w)} \times$$

(*equation continued on p.* 1078)

$$\times [R^{3-w} - r_0^{3-w}] + \frac{B}{\beta R^3} [R - r_0] - \tfrac{1}{2} \Big[\frac{1}{\beta} h_0$$

$$+ \left\{ \frac{1}{R^2} + \frac{1}{R} \left(\frac{\partial r}{\partial t} \right)_s^{-2} \left(\frac{\partial^2 r}{\partial t^2} \right)_s \right\} B \{R - r_0\} \Big]^2 . \qquad \dots(3.7)$$

The energy eqn. (2.7) is evaluated approximately by replacing $\partial r/\partial t$, $\partial r/\partial r_0$ by their values at the shock front (see App. II) while the first integral is evaluated by replacing $p(r_0, t)$ in the integrand by $p(0, t)$ which can be found out from (3.7). Equation (2.7), after integration in this case, reduces to a second order differential equation for R, which after a couple of integrations yields

$$R = Dt^{\delta} \qquad \dots(3.8)$$

where $\delta = \dfrac{2}{5 - w - n}$ and D is a constant.

As in section (2), it may be noted here also that the constant of proportionality D has no effect on the solutions. On plotting the quantities u/u_s, ρ/ρ_s, p/p_s and h/h_s against λ and using (3.8) we get the following relations :

$$\frac{\rho(r_0, t)}{\rho_s(R)} = \frac{p(r_0, t)}{p_s(R)} = \varphi(\xi) \qquad \dots(3.9)$$

$$\rho(r_0, t) = \frac{A}{\beta} \varphi(\xi) R^{-w} \qquad \dots(3.10)$$

$$\frac{h(r_0, t)}{h_0(R)} = 1 + M(1 - \xi) \qquad \dots(3.11)$$

where

$$\varphi(\xi) = 1 + L(1 - \xi^{3-w}) + \frac{1 - \xi}{\beta(1 - \beta) M_h^2} + \frac{1}{2\beta^2(1 - \beta) M_h^2}$$

$$\times [1 - \{1 + M(1 - \xi)\}^2] \qquad \dots(3.12)$$

$$L = \left[1 + \frac{M_h^2 \beta^3}{1 + M_h^2 \beta^2 (1 - 2\beta)} \right] \frac{1}{(3 - w)} \frac{\delta - 1}{\delta}$$

$$+ \frac{M_h^2 \beta^3}{[1 + M_h^2 \beta^2 (1 - 2\beta)]} \frac{1}{(3 - w)} [2 - 2\beta - w]$$

and
$$M = \beta + \frac{\beta}{[1 + M_h^2 \beta^2(1 - 2\beta)]}$$
$$\times \left[\left\{\frac{1 + M_h^2 \beta^2(1 - \beta)}{1 - \beta}\right\} \frac{\delta - 1}{\delta} + \frac{\beta^3 M_h^2 (2 - 2\beta - w)}{1 - \beta}\right].$$

From eqns. (2.1), (3.3), (3.6) and (3.12) we get

$$1 - \lambda_p^3 = 3\beta \int_0^1 \frac{x^{2-w}}{\varphi(x)}\, dx \qquad \text{...(3.13)}$$

$$\lambda^3 - \lambda_p^3 = 3\beta \int_0^\xi \frac{x^{2-w}}{\varphi(x)}\, dx \qquad \text{...(3.14)}$$

and

$$\frac{u(r_0, t)}{u_s(R)} = \frac{1}{(1 - \beta)\,\lambda^2}\left[\lambda_p^3 + \beta \int_0^\xi \frac{1}{\varphi(\xi)} \left\{\frac{\xi}{\rho}\left(\frac{\partial \rho}{\partial \xi}\right)_{t=\text{const}.}\right.\right.$$
$$\left.\left. - \frac{R}{\dot{R}}\,\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_{\xi=\text{const}.}\right\} \xi^{2-w}\, d\xi. \qquad \text{...(3.15)}$$

With the help of eqn. (3.10), eqn. (3.15) can be simplified as

$$\frac{u}{u_s} = \frac{1}{(1 - \beta)\,\lambda^2}\left[\lambda_p^3 + \beta \int_0^\xi \frac{1}{\varphi(x)} \left\{x \frac{\varphi'(x)}{\varphi(x)} + W\right\} x^{2-w}\, dx\right]. \qquad \text{...(3.16)}$$

In general, however, β depends on the shock location (or time). Its value can be determined from the conservation of mass by integrating (2.1) from the piston to the shock front, with the result

$$\int_{\lambda_p}^1 \left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right) = \int_0^1 \left[\frac{\rho_0(r_0)}{\rho_s}\right]\left(\frac{\rho_s}{\rho}\right)\left(\frac{r_0}{R}\right)^2 d\left(\frac{r_0}{R}\right). \qquad \text{...(3.17)}$$

Here ρ_s is the instantaneous density behind the shock which is held constant over the integration. Making use of (1.1), the definition of $\beta = \rho_0/\rho_s$ leads to

$$\beta = \left[\frac{3}{1 - \lambda_p^3} \int_0^1 \left(\frac{\rho_s}{\rho}\right) \xi^{2-w}\, d\xi\right]^{-1}. \qquad \text{...(3.18)}$$

Using the eqn. (3.9) we can evaluate the values of β easily. The values of u/u_s, ρ/ρ_s, p/p_s and h/h_s will be calculated as a function of λ as in section (2).

4. Results and Discussion

We have done the numerical calculations for both the problems choosing $\gamma = 1.4$, $w = 2$, $M_h = 10$ and $n = 0.1$. For strong shock $\beta = (\gamma - 1)/(\gamma + 1)$ in the isothermal case (Laumbach and probstein 1970). One can evaluate the values of β from eqn. (3.18) for more accuracy. We have compared graphically the results obtained in section 2 and 3. The flow variables in the case of adiabatic and isothermal flows have been shown in Figs. 1-4 by smooth and dotted lines respectively. One can, however, calculate the values of flow variables taking different values of γ, w, M_h, n and β. The ranges for w and n are $0 < w < 3/\gamma$ and $0 < n < (3 - w\gamma)$ in adiabatic case and $0 < w < 3/\gamma$ and $0 < n < (3 - w)$ in isothermal case (Ranga Rao and Purohit 1972).

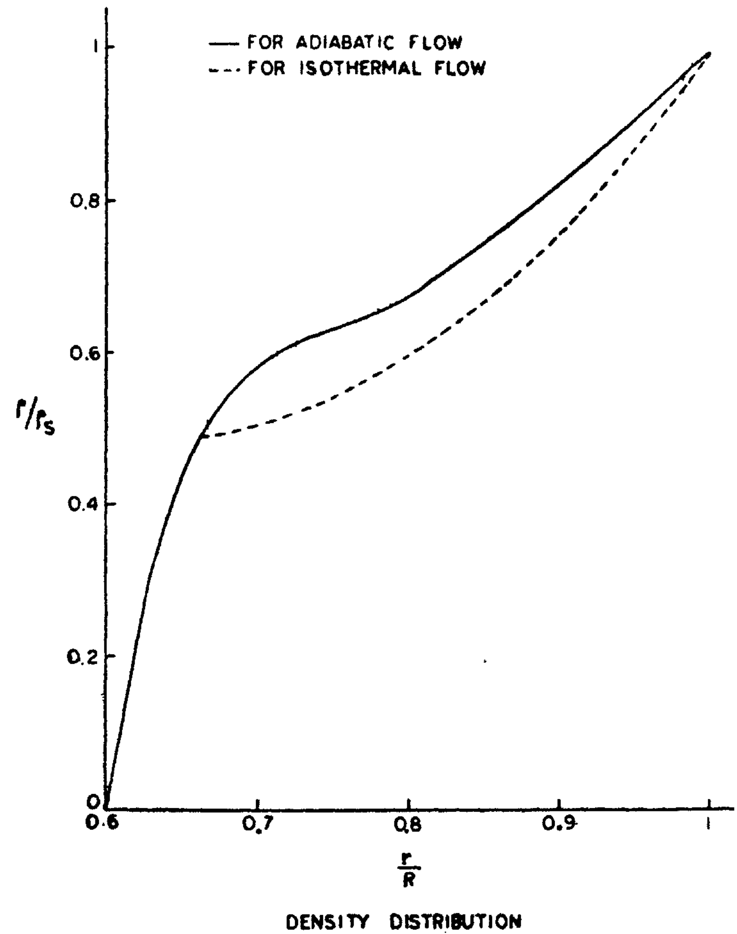
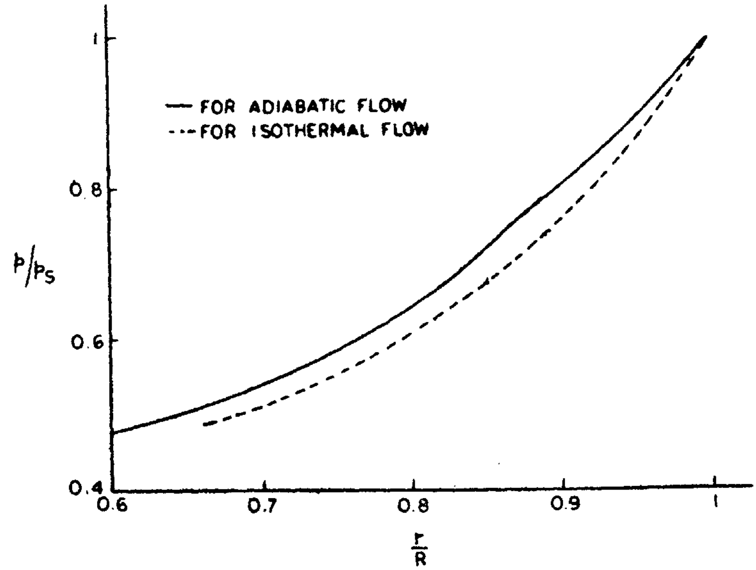
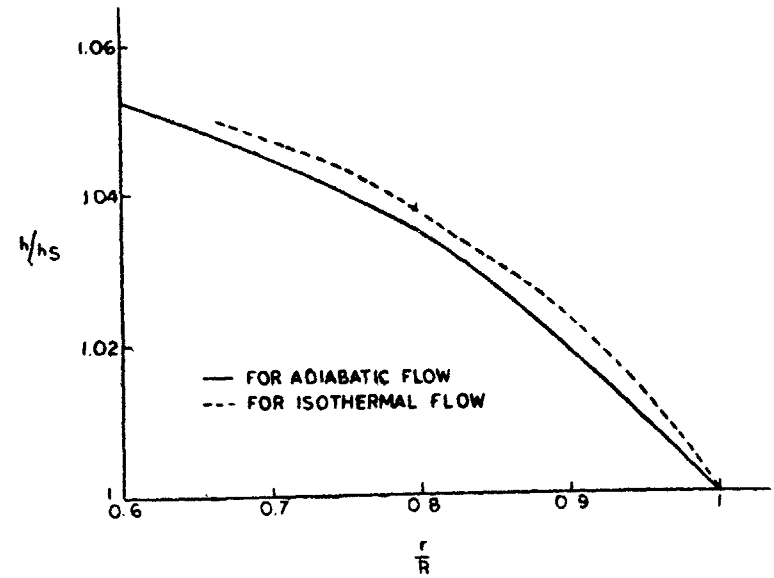


Fig. 1.



PRESSURE DISTRIBUTION

FIG. 2.



MAGNETIC FIELD DISTRIBUTION

FIG. 3.

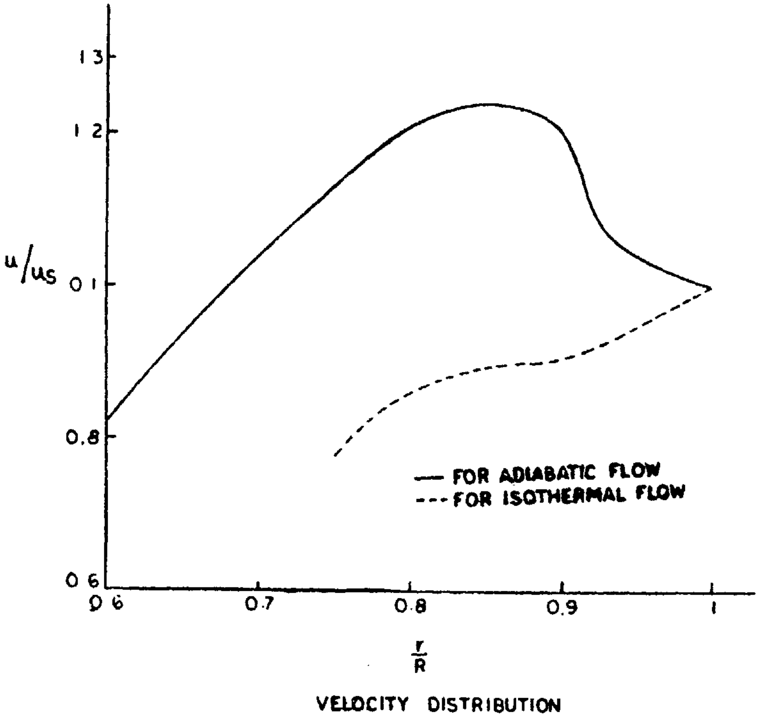


FIG. 4.

REFERENCES

Laumbach, D. D., and Probstein, R. F. (1969). A point explosion in a cold exponential atmosphere. *J. Fluid Mech.*, **35**, 53–75.

———— (1970). Self-similar strong shocks with radiation in decreasing exponential atmosphere. *Phys. Fluids.* **13**, 1178–83.

Ranga Rao, M. P., and Purohit, S. C. (1972). Self-similar flows with increasing energy I & II. *Int. J. Engg. Sci.*, **10**, 249–62; 963–73.

Rosenau, P., and Frankenthal, S. (1976). Equatorial propagation of axisymmetric magnetohydrodynamic shocks. *Phys. Fluids*, **19**, 1889–99.

Summers, D. (1975). An idealised model of a magnetohydrodynamic spherical blast wave applied to a flare produced shock in the solar wind. *Astron. Astrophys.*, **45**, 151–58.

APPENDIX I

We first write the Taylor's expansion for $r(r_0, t)$ as

$$r(r_0, t) = R + \left.\frac{\partial r}{\partial r_0}\right|_R (r_0 - R) + \tfrac{1}{2} \left.\frac{\partial^2 r}{\partial r_0^2}\right|_R (r_0 - R)^2 + \ldots \qquad \ldots(\text{I}.1)$$

From the continuity equation ($\rho_0 r_0^2\, dr_0 = \rho r^2 dr$) we obtain

$$\left.\frac{\partial r}{\partial r_0}\right|_R = \frac{\rho_0}{\rho_s} = \frac{\gamma - 1}{\gamma + 1}, \ (= \beta \text{ for isothermal case}) \qquad \ldots(\text{I}.2)$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial r_0}\bigg|_R = \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\bigg|_R + \frac{4}{\gamma+1}\frac{1}{R} - \frac{\gamma+1}{\gamma-1}\frac{\partial^2 r}{\partial r_0^2}\bigg|_R. \qquad \text{...(I.3)}$$

From the magnetic field equation ($h_0 r_0 dr_0 = hr\,dr$) we obtain

$$\frac{1}{h}\frac{\partial h}{\partial r_0}\bigg|_R = \frac{1}{h_0}\frac{\partial h_0}{\partial r_0}\bigg|_R + \frac{2}{\gamma+1}\frac{1}{R} - \frac{\gamma+1}{\gamma-1}\frac{\partial^2 r}{\partial r_0^2}\bigg|_R. \qquad \text{...(I.4)}$$

By differentiating (I.1) with respect to time and making use of (I.2) we obtain

$$\frac{\partial^2 r}{\partial t^2} = \frac{2}{\gamma+1}\ddot{R} + \frac{\partial^2 r}{\partial r_0^2}\bigg|_R \dot{R}^2 - \frac{\partial^2 r}{\partial r_0^2}\bigg|_R (r_0 - R)\,\ddot{R}. \qquad \text{...(I.5)}$$

From momentum equation we get

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = -\left(\frac{r^2}{\rho_0 r_0^2}\frac{\partial p}{\partial r}\right)\bigg|_R - \left(\frac{r^2}{\rho_0 r_0^2}\,h\,\frac{\partial h}{\partial r_0}\right)\bigg|_R - \frac{h^2}{\rho r}\bigg|_R. \qquad \text{...(I.6)}$$

Equating the values of $(\partial^2 r/\partial t^2)_s$ obtained from (I.5) and (I.6) and also making use of eqns. (2.6) and (2.11) we get

$$-\frac{\partial^2 r}{\partial r_0^2}\bigg|_R \dot{R}^2 = \frac{2}{\gamma+1}\ddot{R} + \frac{2}{\gamma+1}\dot{R}^2\left[\frac{1}{p}\frac{\partial p}{\partial r_0}\right]\bigg|_R + \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-2} \times \frac{1}{h}\frac{\partial h}{\partial r_0}\right]\bigg|_R + \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-1}\frac{1}{r}\right]\bigg|_R. \qquad \text{...(I.7)}$$

Logarithmic differentiation of (2.5) yields

$$\frac{1}{p}\frac{\partial p}{\partial r_0}\bigg|_R = \left(\frac{1}{p_s}\frac{\partial p_s}{\partial r_0} + \frac{\gamma}{\rho}\frac{\partial \rho}{\partial r_0} - \frac{\gamma}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\right)\bigg|_R \qquad \text{...(I.8)}$$

while the shock condition (2.11) gives

$$\frac{1}{p_s}\frac{\partial p_s}{\partial r_0}\bigg|_R = \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r_0}\bigg|_R + \frac{2}{\dot{R}^2}\ddot{R}. \qquad \text{...(I.9)}$$

First, from (I.3), (I.8) and (I.9), calculate $\left(\frac{1}{p}\frac{\partial p}{\partial r_0}\right)\bigg|_R$ and then from (1.1), (1.2), (I.2) and (I.4) calculate

$$\left[\frac{h^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-1}\frac{1}{r}\right]\bigg|_R \text{ and } \left[\frac{h_0^2}{\rho_0}\left(\frac{\partial r}{\partial r_0}\right)^{-2}\frac{1}{h}\frac{\partial h}{\partial r_0}\right]\bigg|_R.$$

Now from (I.5) and (I.7) we finally get the value of $(\partial^2 r/\partial t^2)_s$ as

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \left[\frac{M_h^2}{M_h^2 + \left(\frac{\gamma+1}{\gamma-1}\right)^2}\right]\left[\left\{\frac{4(2\gamma-1)}{(\gamma+1)^2} + \frac{2(\gamma+1)}{(\gamma-1)M_h^2}\right\}\ddot{R} - \left\{\frac{2(\gamma-1)}{(\gamma+1)^3}\langle\, w(\gamma+1) - 4\gamma\,\rangle\right\}\frac{\dot{R}^2}{R}\right]. \qquad \text{...(I.10)}$$

Appendix II

On exactly similar lines just as in appendix I we can derive the value of $(\partial^2 r/\partial t^2)_s$ in the case of isothermal flow as

$$\left(\frac{\partial^2 r}{\partial t^2}\right)_s = \left[1 + \frac{M_h^2 \beta^3}{1 + M_h^2 \beta^2(1 - 2\beta)}\right](1 - \beta)\ddot{R}$$

$$+ \frac{(1 - \beta) M_h^2 \beta^3}{1 + M_h^2 \beta^2(1 - 2\beta)}(2 - 2\beta - w)\frac{\dot{R}^2}{R}. \quad \text{...(II.1)}$$

The only change is that instead of differentiating the eqn. (2.5), we differentiate the equation $p = \Gamma\rho T$ and obtain

$$\frac{1}{p}\left.\frac{\partial p}{\partial r_0}\right|_R = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial r_0}\right|_R \quad \text{...(II.2)}$$

where the condition (3.1) is used.