Indian J. pure appl. Math., 11(8) : 1095–1104, August 1980

WAVE PROPAGATION IN A LINEAR RANDOM NON-HOMOGENEOUS VISCOELASTIC SEMI-INFINITE ROD

GURDARSHAN SINGH AND AVTAR SINGH

Department of Mathematics, Panjabi University, Patiala

(*Received* 19 *April* 1979, *after revision* 20 *February* 1980)

The problem considered in this paper deals with the determination of the mean displacement distribution in a thin linear random non-homogeneous Biot type viscoelastic semi-infinite rod, due to general time-dependent displacement input at the end. A truncated series solution of the wave problem is obtained by using the Laplace transform for small random variations in viscoelastic properties. Three specific cases concerning the probability measure as a function of the continuous type of random variable have been discussed.

1. INTRODUCTION

A random non-homogeneous viscoelastic medium is characterized by a family of relaxation or creep functions depending upon the space coordinates and a parameter α which ranges over a space C in which a probability density $p(\alpha)$ is defined. Of course the relaxation or creep functions must be measurable with respect to $p(\alpha)$. If for each value of α there is determined a unique solution for the displacement $u(x, \alpha, t)$ or for the stress $\sigma(x, \alpha, t)$, then u or σ is a random variable (or function) and its probability density is also $p(\alpha)$. We can then determine the mean wave motion, as well as its variance and other statistics.

In the field of geometrical optics, a number of problems of wave propagation in random media have been solved [see Keller (1962, 1964) and the references contained therein]. Two types of methods are used to solve the wave propagation problems. These are called 'honest' and 'dishonest' methods. In an honest method the solution $u(x, \alpha, t)$ is first determined for each value of α and then its mean value, as well as its variance and other statistics are computed. In the dishonest method randomness is utilized before $u(x, \alpha, t)$ is determined and some unproved assumptions are made about some statistical properties of the random wave motion. In the honest method, more often the solution is expressed in the form of a series in some parameter or it is determined by some other approximation procedure. Keller (1962) has used the perturbation method to study the propagation of light rays in a slightly inhomogeneous random medium.

In the present paper, the mean wave motion in a thin semi-infinite rod of linear randomly non-homogeneous viscoelastic material of the Biot type due to any time-dependent displacement input at the end of the rod, has been determined. The random variations are assumed to be small and the solution of the problem has been obtained by expansion method in the form of a truncated series by using the Laplace transform method. Particular examples concerning the different distributions of the random variable at a particular point of the semi-infinite viscoelastic rod have been discussed.

2. Formulation

The wave problem for the thin semi-infinite rod of random non-homogeneous linear viscoelastic material is formulated by treating the problem as one-dimensional. The end of the rod is taken as $x = 0$ and the coordinate x is measured positive in the direction of the length of the rod, whereas the time is denoted by t. σ, e and u respectively denote the stress, strain and displacement.

Since we consider the simple Biot's model (Nowacki 1962) with a continuous spectrum of relaxation times, the constitutive equation is

$$\sigma = \int_0^t G(x, \alpha, t-\tau) \frac{\partial e}{\partial \tau} d\tau \qquad \text{...(2.1)}$$

where $G(x, \alpha, t) = \lambda(x, \alpha, t) + 2\mu(x, \alpha, t)$, and λ and μ are the usual Lamé's coefficients, taken as functions of x, α and t here. The parameter α ranges over a probability space C in which a probability measure $p(\alpha)$ is defined.

The equation of motion and the strain-displacement relation are

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \text{...(2.2)}$$

$$e = \frac{\partial u}{\partial x} \qquad \text{...(2.3)}$$

where the density ρ is the initial density taken as constant.

Introducing the displacement function ϕ through the equation

$$u = \frac{\partial \phi}{\partial x} \qquad \text{...(2.4)}$$

eqns. (2.1), (2.2) and (2.3) lead to

$$\int_0^t G(x, \alpha, t-\tau) \frac{\partial^3 \phi}{\partial x^2 \partial \tau} \partial\tau - \rho \frac{\partial^2 \phi}{\partial t^2} = 0. \qquad \text{...(2.5)}$$

Now applying the Laplace transform on eqn. (2.5) and assuming the quiescent initial conditions

$$\phi_{t=0} = (\partial\phi/\partial t)_{t=0} = 0 \qquad ...(2.6)$$

we obtain

$$p\bar{G}(x, \alpha, p)\frac{\partial^2\bar{\phi}}{\partial x^2} - \rho p^2\bar{\phi} = 0 \qquad ...(2.7)$$

where p is the transform parameter and a superimposed bar indicates Laplace transform.

3. Approximate Solution of Equation (2.7) as a Truncated Series

We consider the special case, when

$$G(x, \alpha, t) = (G_0 + \epsilon g(x, \alpha))\, e^{-\eta t}, \qquad ...(3.1)$$

where $g(x, \alpha)$ is a bounded function of x and α. The constant ϵ is assumed to be small and the relaxation parameter $\eta > 0$ (Nowacki 1962). In this case we have

$$\bar{G}(x, \alpha, p) = G_0\left(1 + \frac{\epsilon}{G_0}g(x, \alpha)\right)\Big/(p + \eta). \qquad ...(3.2)$$

Assume the solution of eqn. (2.7) as

$$\bar{\phi} = \bar{\phi}_0 + (\epsilon/G_0)\,\bar{\phi}_1 + (\epsilon/G_0)^2\,\bar{\phi}_2 + \ldots \qquad ...(3.3)$$

where ϵ/G_0 is regarded as small.

Upon inserting (3.2) and (3.3) into (2.7), and equating to zero the coefficient of each power of ϵ/G_0, we get

$$\bar{\phi}_{0,xx} - \frac{p(p+\eta)}{c^2}\bar{\phi}_0 = 0 \qquad ...(3.4)$$

and

$$\bar{\phi}_{n,xx} - \frac{p(p+\eta)}{c^2}\bar{\phi}_n = -\bar{\phi}_{n-1,xx}\,g(x, \alpha),\ n = 1, 2, 3, \ldots \qquad ...(3.5)$$

where $c = (G_0/\rho)^{1/2}$ is the longitudinal wave speed in the homogeneous case. Also we regard $p(p + \eta)$ as positive as in the elastic case ($\eta = 0$).

Solution of equation (3.4) which is bounded as $x \to \infty$, is

$$\bar{\phi}_0 = A(p)\exp\left(-\sqrt{p(p+\eta)}\,x/c\right). \qquad ...(3.6)$$

To evaluate $A(p)$, we use the boundary condition

$$u(0, t) = f(t), \qquad ...(3.7)$$

where $f(t)$ is a suitably prescribed function of t. This implies that

$$\bar{u}(0, p) = \bar{\phi}_{,x}(0, p) = \bar{f}(p).$$

This condition will be satisfied if

$$\bar{\phi}_{0,x}(0, p) = \bar{f}(p),\ \bar{\phi}_{n,x}(0, p) = 0 \text{ for } n = 1, 2, 3, \ldots \qquad \ldots(3.8)$$

Using the first of these and (3.6) to obtain $A(p)$ and then substituting it back in (3.6), $\bar{\phi}_{0,x}(x, p)$ is obtained as

$$\bar{\phi}_{0,x}(x, p) = \bar{f}(p) \exp\left(-\sqrt{p(p+\eta)}\, x/c\right). \qquad \ldots(3.9)$$

Now by using the convolution theorem, we get from (3.9)

$$\phi_{0,x}(x, t) = \int_0^t f(t-\tau)\, P(x, \tau)\, d\tau, \qquad \ldots(3.10)$$

where

$$P(x, t) = L^{-1}\left[\exp\left(-\sqrt{p(p+\eta)}\, x/c\right)\right]$$

and L^{-1} denotes the inverse Laplace transform.

Making use of suitable standard results for Laplace inverse transform, it can be shown that

$$P(x, t) = e^{-\eta t/2}\left[\frac{\eta x}{2c}\, \frac{I_1(\eta/2\,[t^2 - (x/c)^2]^{1/2})}{[t^2 - (x/c)^2]^{1/2}}\, H(t - x/c) + I_0\left(\frac{\eta}{2}\,[t^2 - (x/c)^2]^{1/2}\right) \delta(t - x/c)\right] \qquad \ldots(3.11)$$

where $H(x)$ is the Heaviside unit function, $\delta(x)$ is Dirac's delta function and $I_0(x)$, $I_1(x)$ are the usual Bessel's functions.

Substituting (3.11) in (3.10), we obtain

$$\left.\begin{aligned} \phi_{0,x}(x, t) &= 0, && \text{for } x > ct \\ &= R(x, t), && \text{for } 0 < x < ct \end{aligned}\right\} \qquad \ldots(3.12)$$

where

$$R(x, t) = f(t - x/c)\, e^{-\eta x/2c} + \frac{\eta x}{2c}\int_{x/c}^{t} f(t-\tau)\, e^{-\eta\tau/2} \times \frac{I_1(\eta/2\,[\tau^2 - (x/c)^2]^{1/2})}{[\tau^2 - (x/c)^2]^{1/2}}\, d\tau. \qquad \ldots(3.13)$$

This function represents the solution for the homogeneous non-random viscoelastic rod of Biot type.

Next solving the differential equation for $\bar{\phi}_1$ with the boundary and regularity conditions

$$\phi_{1,x}(0, \alpha, t) = 0$$

$$\phi_1(x, \alpha, t) \to 0 \text{ as } x \to \infty \qquad \ldots(3.14)$$

we obtain, after using the convolution theorem and some lengthy calculations

$$\phi_{1,x}(x, \alpha, t) = 0 \quad \text{for } x > ct \qquad \ldots(3.15)$$

$$= \frac{1}{2c}\int_{x/c}^{t} f(t-s)\, e^{-\eta s/2}\Big[\tfrac{1}{4}\int_{x}^{cs}\Big\{\frac{\eta^2 X^2}{2c^2(s^2 - X^2/c^2)} I_0(\eta/2\,[s^2 - (X/c)^2]^{1/2})$$

$$- \frac{\eta(s^2 + X^2/c^2)}{(s^2 - X^2/c^2)^{3/2}} I_1(\eta/2\,[s^2 - (X/c)^2]^{1/2})\Big\}$$

$$\times \Big\{g\Big(\frac{X-x}{2}, \alpha\Big) - g\Big(\frac{X+x}{2}, \alpha\Big)\Big\}\, dX$$

$$+ \tfrac{1}{2}\Big\{\frac{\eta^2 X^2 I_0(\eta/2\,[s^2 - (x/c)^2]^{1/2})}{2c^2(s^2 - X^2/c^2)} - \frac{\eta(s^2 + x^2/c^2)}{(s^2 - x^2/c^2)^{3/2}}$$

$$\times I_1(\eta/2\,[s^2 - (x/c)^2]^{1/2})\, g_1(x, \alpha)$$

$$+ \frac{\eta^2 cs}{16}\Big\{g\Big(\frac{cs-x}{2}, \alpha\Big) - g\Big(\frac{cs+x}{2}, \alpha\Big)$$

$$- \frac{c^2}{4}\Big(g'\Big(\frac{cs-x}{2}, \alpha\Big) - g'\Big(\frac{cs+x}{2}, \alpha\Big)\Big)\Big\}\Big]\, ds$$

$$+ \tfrac{1}{4} f(t - x/c)\, e^{-\eta x/2c}\Big[g(0, \alpha) - g(x, \alpha) + \frac{\eta}{c}\Big(\frac{\eta x}{4c} + 1\Big) g_1(x, \alpha)\Big]$$

$$+ \frac{1}{2c} f'(t - x/c)\, e^{-\eta x/2c}\, g_1(x, \alpha) \quad \text{for } 0 < x < ct$$

where

$$g_1(x, \alpha) = \int_0^x g(s, \alpha)\, ds \qquad \ldots(3.16)$$

and the 'dash' denotes derivative w.r.t. the sure (i.e. non-random) argument of the function concerned.

Similarly we can find $\phi_{2,x}$, $\phi_{3,x}$, ... and the solution for the displacement $u(x, \alpha, t)$ is determined for each value of α from (3.3) as (assuming the absolute and uniform convergence of the series involved)

$$u(x, \alpha, t) = \phi_{0,x}(x, t) + (\epsilon/G_0)\, \phi_{1,x}(x, \alpha, t) + (\epsilon/G_0)^2\, \phi_{2,x}(x, \alpha, t) + \ldots \qquad \ldots(3.17)$$

The randomness in the above process plays no role and may have the useful effect of yielding simpler expressions for the statistics of u than those for u itself. If we specify the probability space C, the probability density $p(\alpha)$ and $g(x, \alpha)$, we can compute the mean value of u which is just the sum of the mean values of the terms on the right side of (3.17). The mean value of any function $f(\alpha)$ is defined by

$$\langle f \rangle = \int_C f(\alpha)\, p(\alpha)\, d\alpha. \qquad \text{...(3.18)}$$

From eqn. (3.12), since $\phi_{0,x}$ is independent of $g(x, \alpha)$ and therefore of α, we have

$$\langle \phi_{0,x} \rangle = \phi_{0,x}. \qquad \text{...(3.19)}$$

From eqn. (3.15), by interchanging the order of taking the mean value with integration and differentiation, which we assume to be permissible, we obtain

$$\langle \phi_{1,x} \rangle = 0, \quad \text{for } x > ct$$

$$= \frac{1}{2c} \int_{x/c}^{t} f(t - s)\, e^{-\eta s/2} \Bigg[\frac{1}{4} \int_{x}^{cs} \Bigg\{ \frac{\eta^2 X^2}{2c^2(s^2 - X^2/c^2)} \times I_0(\eta/2\, [s^2 - (X/c)^2]^{1/2})$$

$$- \frac{\eta(s^2 + X^2/c^2)}{(s^2 - X^2/c^2)^{3/2}}\, I_1(\eta/2\, [s^2 - (X/c)^2]^{1/2}) \Bigg\}$$

$$\times \left\{ \left\langle g\left(\frac{X - x}{2}, \alpha\right)\right\rangle - \left\langle g\left(\frac{X + x}{2}, \alpha\right)\right\rangle \right\} dX$$

$$+ \tfrac{1}{2} \Bigg\{ \frac{\eta^2 x^2 I_0(\eta/2\, [s^2 - (x/c)^2]^{1/2})}{2c^2(s^2 - x^2/c^2)}$$

$$- \frac{\eta(s^2 + x^2/c^2)\, I_1(\eta/2\, [s^2 - (x/c)^2]^{1/2})}{(s^2 - x^2/c^2)^{3/2}} \Bigg\} \langle g_1(x, \alpha) \rangle$$

$$+ \frac{\eta^2 cs}{16} \Bigg\{ \left\langle g\left(\frac{cs - x}{2}, \alpha\right)\right\rangle - \left\langle g\left(\frac{cs + x}{2}, \alpha\right)\right\rangle$$

$$- \frac{c^2}{4} \left(\left\langle g'\left(\frac{cs - x}{2}, \alpha\right)\right\rangle - \left\langle g'\left(\frac{cs + x}{2}, \alpha\right)\right\rangle \right) \Bigg\} \Bigg] ds$$

$$+ \tfrac{1}{4} f(t - x/c)\, e^{-\eta x/2c} \Bigg[\langle g(0, \alpha) \rangle - \langle g(x, \alpha) \rangle$$

$$+ \frac{\eta}{c} \left(\frac{\eta x}{4c} + 1 \right) \langle g_1(x, \alpha) \rangle \Bigg]$$

$$+ \frac{1}{2c} f'(t - x/c)\, e^{-\eta x/2c} \langle g_1(x, \alpha) \rangle, \text{ for } 0 < x < ct. \qquad \text{...(3.20)}$$

The computation of the mean value of $\phi_{2,x}$ will involve the mean value of certain quadratic expressions in g and its derivatives and so on.

4. Particular Cases

Example 1 — We compute the mean value of the displacement distribution in the rod by considering the random non-homogeneous function of the following form :

$$g(x, \alpha) = h(x)\, e^{-k\alpha} \qquad \text{...(4.1)}$$

where k is a positive constant, $h(x)$ a bounded function of x and the distribution of the random variable α is given by the probability density function of the gamma-type as

$$p(\alpha) = \frac{1}{\Gamma(a)\, b^a}\, \alpha^{a-1} e^{-\alpha/b}, \quad 0 < \alpha < \infty$$

$$= 0, \text{ elsewhere} \qquad \text{...(4.2)}$$

where a and b are positive constants.

In this case we obtain the solution for the mean value of displacement in the form of a series upto the first power of (ϵ/G_0), as

$$\langle\, u \,\rangle = 0, \text{ for } x > ct,$$

$$= R(x, t) + (\epsilon/G_0)\, M \left[\frac{1}{2c} \int_{x/c}^{t} f(t - s)\, e^{-\eta s/2} \right.$$

$$\times \left\{ \frac{1}{2} \int_{x}^{cs} \left(\frac{\eta^2 X^2 I_0(\eta/2\,[s^2 - (X/c)^2]^{1/2})}{2c^2(s^2 - X^2/c^2)} \right. \right.$$

$$- \frac{\eta(s^2 + X^2/c^2)}{(s^2 - X^2/c^2)^{3/2}}\, I_1\left(\frac{\eta}{2} \sqrt{s^2 - (X/c)^2} \right)$$

$$\times \left(h\left(\frac{X - x}{2}\right) - h\left(\frac{X + x}{2}\right)\right) dX$$

$$+ \frac{1}{2} \left(\frac{\eta^2 x^2 I_0(\eta/2\,[s^2 - (x/c)^2]^{1/2})}{2c^2(s^2 - x^2/c^2)} \right.$$

$$\left. - \frac{\eta(s^2 + x^2/c^2)\, I_1(\eta/2\,[s^2 - (x/c)^2]^{1/2})}{(s^2 - x^2/c^2)^{3/2}} \right) h_1(x)$$

$$+ \frac{\eta^2 cs}{16} \left[h\left(\frac{cs - x}{2}\right) - h\left(\frac{cs + x}{2}\right) \right.$$

$$\left. \left. - \frac{c^2}{4} \left(h'\left(\frac{cs - x}{2}\right) - h'\left(\frac{cs + x}{2}\right)\right)\right]\right\} ds +$$

(*equation continued on p.* 1102)

$$+ \tfrac{1}{4} f(t - x/c)\, e^{-\eta x/2c} \left\{ h(0) - h(x) + \frac{\eta}{c} \left(\frac{\eta x}{4c} + 1 \right) h_1(x) \right\}$$

$$+ \frac{1}{2c} f'(t - x/c)\, e^{-\eta x/2c} h_1(x) \Big], \text{ for } 0 < x < ct. \qquad \ldots(4.3)$$

From the resulting expression (4.3) we observe that the effect of the continuous random medium represented by eqns. (4.1) and (4.2) on the propagation of waves in the rod, is that the non-homogeneous part of the mean displacement distribution along the rod is multiplied by a constant $M = 1/(1 + kb)^a$.

Example 2 — Let us now prescribe $g(x, \alpha)$ as

$$g(x, \alpha) = h(x) + k\alpha, \qquad \ldots(4.4)$$

where k is a finite constant, $h(x)$ a bounded function of x and the distribution of the random variable α is given by the probability density

$$p(\alpha) = \tfrac{1}{2} (\alpha + 1), \ -1 < \alpha < 1$$

$$= 0, \text{ elsewhere}. \qquad \ldots(4.5)$$

The approximate solution for the mean displacement distribution in this case takes the following form :

$$\langle u \rangle = 0, \text{ for } x > ct$$

$$= R(x, t) + (\epsilon/G_0) \left[\frac{1}{2c} \int_{x/c}^{t} f(t - s)\, e^{-\eta s/2} \right.$$

$$\times \left\{ \tfrac{1}{4} \int_{x}^{cs} \left(\frac{\eta^2 X^2 I_0(\eta/2\, [s^2 - (X/c)^2]^{1/2})}{2c^2(s^2 - X^2/c^2)} \right. \right.$$

$$- \frac{\eta(s^2 + X^2/c^2)}{(s^2 - X^2/c^2)^{3/2}} I_1 \left(\frac{\eta}{2} \sqrt{s^2 - (X/c)^2} \right) \Big)$$

$$\times \left(h\left(\frac{X - x}{2}\right) - h\left(\frac{X + x}{2}\right) \right) dX$$

$$+ \tfrac{1}{2} \left(\frac{\eta^2 x^2}{2c^2(s^2 - x^2/c^2)} \right) I_0 \left(\frac{\eta}{2} \sqrt{s^2 - (x/c)^2} \right)$$

$$- \frac{\eta(s^2 + x^2/c^2)}{(s^2 - x^2/c^2)^{3/2}} I_1 \left(\frac{\eta}{2} \sqrt{s^2 - (x/c)^2} \right) \Big)$$

$$\times \left(h_1(x) + \frac{kx}{3} \right) + \frac{\eta^2 cs}{16} \left(h\left(\frac{cs - x}{2}\right) - h\left(\frac{cs + x}{2}\right) - \right.$$

(equation continued on p. 1103)

$$- \frac{c^2}{4} \left(h'\left(\frac{cs - x}{2}\right) - h'\left(\frac{cs + x}{2}\right)\right)\Big)\Big\} \, ds$$

$$+ \tfrac{1}{4} f(t - x/c)\, e^{-\eta x/2c} \left\{h(0) - h(x) + \frac{\eta}{c}\left(\frac{\eta x}{4c} + 1\right)\left(h_1(x) + \frac{kx}{3}\right)\right\}$$

$$+ \frac{1}{2c} f'(t - x/c)\, e^{-\eta x/2c}\left(h_1(x) + \frac{kx}{3}\right)\Big] \text{ for } 0 < x < ct. \quad ...(4.6)$$

In this case, from expression (4.6) we note that due to the random variation of the viscoelastic properties of the medium considered, the magnitude of the mean displacement in the non-homogeneous viscoelastic rod increases for $k > 0$ and decreases for $k < 0$, provided the coefficient of k in the first term involving k, is positive.

Example 3.1 — Here we set

$$g(x, \alpha) = h(x) \,.\, k\alpha \quad ...(4.7)$$

where k is a finite constant, $h(x)$ a bounded function of x and the distribution of the random variable α is given by the probability density

$$p(\alpha) = \tfrac{1}{2}, \quad -1 < \alpha < 1$$
$$= 0, \text{ elsewhere.} \quad ...(4.8)$$

In this case the mean displacement distribution is given by (3.12). Therefore, for the particular case (4.7) in which the random variable α has the constant distribution over the interval $(-1, 1)$, the non-homogeneous part of the mean displacement distribution in the rod becomes zero and the mean displacement is obtained as in the homogeneous viscoelastic case.

Example 3.2 — In this case we prescribe the non-homogeneous random function $g(x, \alpha)$ by (4.7), where the distribution of the random variable α is given by the probability density

$$p(\alpha) = 2(1 - \alpha), \quad 0 < \alpha < 1$$
$$= 0, \text{ elsewhere.} \quad ...(4.9)$$

The mean displacement distribution in this case takes the form (4.3) with the value of the constant $M = k/3$.

5. Concluding Remarks

The meam displacement and mean stress at any point of the randomly non-homogeneous rod can be determined in the same way as above if instead of the displacement, the velocity or the stress is prescribed at the end of the rod. It may be noted that whereas the boundary condition is deterministic, the physical properties of the rod are of a random nature, with only slight variations. If the variations are not so small the present series expansion method is not likely to be very reliable. Other interesting permutations and combinations are obviously possible. In this connection we may refer to a general theory of statistical mechanics of viscoelasticity given by Devault and Mclennan (1965). They incorporate their theory of viscoelasticity for materials of large internal relaxation times from a consideration of statistical mechanics, employing the theory of linear transport excluding thermal conduction.

Similarly Gabrielsen (1968) considers stochastic models for viscoelastic materials, developing a stochastic analog to describe the creep phase of mechanical viscoelastic models based on the microbehavior of structural materials, such as soil and polymer plastics.

The problem of random processes of change in loading and temperature in predicting the creep of polymeric materials was considered by Anrikson *et al.* (1977). Another interesting work in this connection is by Balakrishnan *et al.* (1978) who consider a stochastic theory of anelastic creep.

For the corresponding problem of non-random non-homogeneous media reference may be made to Krickaja (1971) and Singh and Singh (1974).

Acknowledgement

The authors are thankful to the referee for suggesting certain improvements in the paper. One of the authors (G. S.) expresses his thanks to CSIR for granting him a Senior Research Fellowship.

References

Anrikson, G. A., Kalnroze, Z. V., and Urzhumstev, Yu. S. (1977). Prediction of Creep of Polymeric Materials During Random Precesses of Change in Loading and Temperature-Humidity Conditions of Surrounding Media : Part 2, Polymer Mechanics, **12**, 4, pp. 552–57, (*Trans. Mekhnika Polimerov*, **4**, (July/Aug. 1976), pp. 616–21, by Consultants Bureau, New York).

Balakrishnan, V., Dattagupta, S., and Venkataraman, G. (1978). A stochastic theory of anelastic creep. *Phil. Mag.*, **37**, 65–84.

Devault, G. P., and Mclennan, J. A. (1965). Statistical mechanics of viscoelasticity. *Phys. Rev.*, **137**, 3A, 724–30.

Gabrielsen, B. L. (1968). Stochastic models for viscoelastic materials. *Materials and Structures*, **1**, 319–26.

Keller, J. B. (1962). Wave Propagation in Random Media. Proc. of Sym. in Appl. Math., Vol. 13, Am. Math. Soc., New York pp. 227–46.

——— (1964). Stochastic Equations and Wave Propagation in Random Media. Proc. of Sym. in Appl. Math., Vol. 16, Amer. Math. Soc., New York, pp. 145–70.

Krickaja, S. S. (1971). Mathematical consideration of the problem of the longitudinal oscillations of an elasto-viscous non-homogeneous rod. *Isv. Vyss. Ucebn. Zaved, Matematika*, **4**, 107, 37–41.

Nowacki, W. (1962). Thermoelasticity. Pergamon Press, London pp. 550, 558.

Singh, A., and Singh, G. (1974). Wave Propagation in a linear non-homogeneous thin viscoelastic semi-infinite rod. *Utilitas Mathematica*, **5**, 123–35.