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MHD FLOW THROUGH A CHANNEL OF VARYING GAP

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To understand the effect of magnetic field on the abnormal flow conditions caused by the presence of stenosis in arteries, analytical solutions are obtained for the steady laminar conducting flow of an incompressible Newtonian fluid in an axisymmetric channel of varying gap. Three approximation methods are developed depending upon the geometrical configuration. The results obtained are applied to study the flow of a conducting fluid through a smooth constriction. It is observed that the overall effect of magnetic field is to decrease the resistance to flow and shear stress at the wall and to reduce the abnormalities of flow due to irregular boundaries.

1. INTRODUCTION

It is well known (Young 1968) that at various locations in the arterial system, stenosis may develop due to abnormal intravascular growths. For example, arteries may be narrowed by the development of atherosclerotic plaques which is closely connected to the hydrodynamics of blood flow through the artery. The study of the hydrodynamic aspect of the blood flow shows (Langlois 1958, Lee and Fung 1970, Chow and Soda 1972) that the stenosis developed in the artery causes the abnormal flow which is an important factor in the development and progression of arterial disease. Although the specific reason for the initiation of growth, which eventually projects into the lumen of the artery, is not known, it is clear that if such an event occurs the flow characteristics in the vicinity of the resulting protuberance may significantly be altered. The important flow characteristics in the arterial system are the pressure, shear stress and possible change in the flow characteristics. These, in the presence of stenosis, are related respectively to the physiologically important problems of: increase in the resistance to the blood flow; possible damages to the red and endothelial cells due to the existence of high shear zones; and possible transition from laminar to turbulent flow inside the blood vessel creating high intensity shear zones unfavourable to the blood and arterial wall.

Previous works (Langlois 1958, Young 1968, Lee and Fung 1970, Chow and Soda 1972) on the study of flow through irregular boundaries are connected only with the hydrodynamic aspect of flow. But no literature is available, to the authors' knowledge, on the study of blood flow in a vessel with an irregular surface in the presence of an electromagnetic field. A human system, particularly the blood flow, is influenced by the external electromagnetic fields. The effects of electromagnetic force, i.e., Lorentz force, on the flow of blood in the presence of a stenosis are:

(a) To decrease resistance to flow with possible increase in blood flow which creates high pressure regions in the areas of low pressure regions. In other words the Lorentz force avoids the danger of complete occlusion.

(b) To reduce the high shear stress caused by stenosis and hence to prevent the damage to the red and endothelial cells.

(c) To delay the transition from laminar to turbulent flow (Chandrasekhar 1961, Rudraiah 1964a, b, 1970) inside the blood vessel and thus reducing high intensity shear zones which are unfavourable to the blood and arterial wall.

Therefore, the effect of electromagnetic field on the flow in a vessel with irregular surface is favourable in understanding and prevention of arterial diseases. With this motivation in mind, we investigate here the flow of a viscous conducting fluid in a channel of varying gap in the presence of a magnetic field.

In this study, to obtain analytical solutions, three approximate methods are developed for three different situations. In the first method, wall slope is assumed to be negligible and the results obtained are similar to those of Hartmann flow. In the second method wall curvature is assumed to be negligible and the analysis is carried out by approximating the channel to that of a divergent wedge with a source at its vertex. In the third method, the results of the second method are expanded in power series in terms of the wall slope and the results are applied to the problems of flow in a channel with constriction. The results confirm the effects predicted in (a), (b) and (c) above.

2. Formulation of the Problem

The general problem of flow of blood through arteries (i.e. irregular boundaries) is mathematically complicated due to non-Newtonian nature of blood through irregular geometries. Because of the complexity of this problem for realistic flow configurations, the study of simplified flow models is helpful in illuminating some of the major physical features involved in the interaction of magnetic field with flow characteristics induced by a stenosis (i.e. irregular boundary). An idealized model for this purpose is an infinitely long channel of smoothly varying gap (Fig. 1) through which laminar steady highly viscous conducting fluid flows for which closed form solutions are possible. We assume that blood behaves like a homogeneous conducting Newtonian fluid with constant density ρ, viscosity μ_f and electrical conductivity σ.

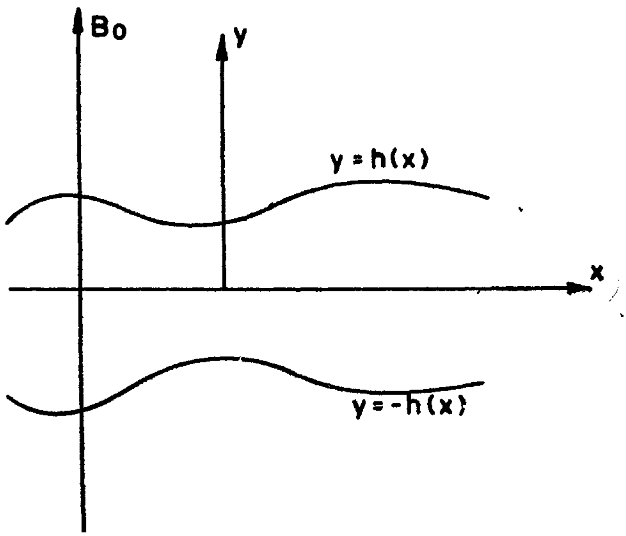

FIG. 1. Physical model.

It is further assumed that the channel has symmetry along the x-axis. It is of interest to note that in actual problem of flow of blood through arteries, in general, complex three-dimensional flow patterns are developed near the stenosis which are virtually impossible to analyse. In some instances, the stenosis is known to be more 'collar like' (Young 1968) with some degree of axial symmetry. Therefore, the assumption of axial symmetry in the present paper is reasonably valid. A uniform magnetic field B_0 is applied in the y direction. Suppose the walls of the channel be represented by curves

$$y = \pm h(x) \qquad \text{...(2.1)}$$

where $h(x)$ is continuous and positive for all x.

The equations of motions, neglecting inertia (for we are considering highly viscous fluid) and induced magnetic field (i.e. we assume that the magnetic Reynolds number is very small which is usual in the case of blood flow) effects, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{...(2.2)}$$

$$\nabla^2 u - \frac{\sigma B_0^2}{\mu_f} u = \frac{1}{\mu_f} \frac{\partial p}{\partial x} \qquad \text{...(2.3)}$$

$$\nabla^2 v = \frac{1}{\mu_f} \frac{\partial}{\partial y} (p + \tfrac{1}{2} B_x^2) \qquad \text{...(2.4)}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(u, v) are the components of velocity in the x- and y-directions, B_0 is the applied magnetic field in the y-direction, B_x the induced magnetic field in the x-direction, and p the pressure. The boundary conditions are the no-slip boundary conditions

$$u(x, \pm h(x)) = v(x, \pm h(x)) = 0. \quad \ldots(2.5)$$

Eliminating the pressure between eqns. (2.3) and (2.4) we have third order differential equation for u and v. Hence in addition to the no-slip boundary condition (2.5) we need one more boundary condition. This boundary condition is obtained by calculating the flux across the channel which has to be constant at all cross-sections of the channel for an incompressible fluid. Hence

$$\int_{-h(x)}^{h(x)} u\, dy = Q \text{ for all } x \quad \ldots(2.6)$$

where Q is the net flow through the channel.

3. Solutions of the Problem

To find the solution of the problem three approximate methods depending upon the three physical situations are developed and these are discussed in this section.

First Approximation Method : Wall Slope Everywhere Negligible

If the wall slope $h'(x)$ is everywhere small compared to unity, it is reasonable to assume that at each value of x, the components of velocity and pressure gradient are approximately equal to those obtained in the case of uniform channel flow. This approximation leads to Hartmann velocity distribution with a pressure gradient parallel to the axis of the channel. Under this approximation the basic equations are

$$\frac{\partial}{\partial \xi}(p + \tfrac{1}{2} B_x^2) = 0 \quad \ldots(3.1)$$

$$v = 0 \quad \ldots(3.2)$$

$$\frac{\partial p}{\partial x} \frac{h^2}{\mu_f} = \frac{\partial^2 u}{\partial \xi^2} - M^2 u \quad \ldots(3.3)$$

where $y = h\xi$ and $M^2 = \dfrac{B_0^2 h^2 \sigma}{\mu_f}$ is the square of the Hartmann number.

The solution of (3.3) using eqns. (2.5) and (2.6) is

$$u = \frac{QM}{2h}\left[\frac{\cosh(My/h) - \cosh M}{\sinh M - M\cosh M}\right] \qquad \text{...(3.4)}$$

where

$$Q = 2h\left[\frac{\sinh M - M\cosh M}{M\cosh M}\right]\frac{h^2}{M^2\mu_f}\frac{\partial p}{\partial x}. \qquad \text{...(3.5)}$$

The expression for pressure is

$$\frac{P}{Q\mu_f} = \int_c^x \frac{M^3\cosh M}{2(\sinh M - M\cosh M)}\frac{dx}{h^3} \qquad \text{...(3.6)}$$

where c is a constant of integration. Equations (3.4) and (3.6) in the limit $M \to 0$, become

$$u = \frac{3Q}{4h}\left(1 - \frac{y^2}{h^2}\right) \qquad \text{...(3.7)}$$

and

$$\frac{\partial p}{\partial x} = -\frac{3}{2}\frac{Q\mu_f}{h^3} \qquad \text{...(3.8)}$$

which are the usual hydrodynamic equations. Equations (3.2) and (3.4) satisfy the boundary conditions exactly. Although eqns. (3.4) to (3.6) are similar to the Hartmann flow solutions, they differ from them in the sense that in the present analysis u and p are functions of both x and y. The components of velocity, given by eqns. (3.2) and (3.4) are the possible components, if and only if

$$\left.\begin{aligned} &|h'| \ll 1 \\ &\left|h(x)\,h'(x)\cosh\left(\frac{My}{h}\right)\right| \ll 1 \\ &\left|h'(x)\,My\sinh\left(\frac{My}{h}\right)\right| \ll 1. \end{aligned}\right\} \qquad \text{...(3.9)}$$

Equation (3.9) implies that for eqns. (3.4) to (3.6) to be valid both wall slope and Hartmann number must be small.

Second Approximation Method : Wall Curvature Everywhere Negligible

We have discussed above the solution for the case where the wall slope is negligible i.e. $h'(x) \ll 1$. This restriction can be removed by assuming that the flow locally to be as if $h(x)$ were a linear function of x. Depending on the sign of $h'(x)$ we have different geometrical situations of the channel. If $h'(x)$ is positive, the channel is approximated by a divergent wedge with a source of flux Q at its vertex. If $h'(x)$ is negative, the wedge is convergent with a sink at its vertex. The analysis is carried out assuming $h'(x)$ as positive and similar results can be obtained when $h'(x)$ is negative.

The equations of motion for the creeping flow in cylindrical co-ordinates (r, θ, z) with an applied magnetic field in the θ-direction are

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta} = 0 \qquad ...(3.10)$$

$$\frac{\partial p}{\partial r} = \mu_f\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) - \sigma B_\theta^2 u \qquad ...(3.11)$$

$$\frac{\partial p}{\partial \theta} = r\mu_f\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right) + \sigma B_r B_\theta u \qquad ...(3.12)$$

where u and v are the components of velocity in r and θ directions respectively. B_θ is the applied magnetic field in θ-direction and B_r is the induced magnetic field in the radial direction, and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \qquad ...(3.13)$$

The boundary conditions are

$$\left.\begin{array}{l} u(r, \pm\alpha) = v(r, \pm\alpha) = 0 \\ \int_{-\alpha}^{\alpha} ru\, d\theta = Q. \end{array}\right\} \qquad ...(3.14)$$

We assume $B_\theta = (A/r)$ where A is a constant to be determined by the strength of an isolated current flowing along the z-axis. Before finding the solutions of the problem, we make (3.10) to (3.13) dimensionless using the quantities

$$p^* = \frac{p}{\rho\omega^2},\ u^* = \frac{u}{\omega},\ v^* = \frac{v}{\omega},\ r^* = \frac{r}{\beta}$$

$$Q^* = Q/\beta\omega$$

where the asterisks denote the dimensionless quantities. For simplicity neglecting the asterisks, we have (for $R_m \ll 1$)

$$R\frac{\partial p}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2}(1 + M^2) - \frac{2}{r^2}\frac{\partial v}{\partial \theta} \qquad ...(3.15)$$

$$\frac{R}{r}\frac{\partial p}{\partial \theta} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} \qquad ...(3.16)$$

where $R = \omega\beta/\nu$ is the Reynolds number, $M^2 = (\sigma B_0^2\beta^2/\rho\nu)$ is the square of the Hartmann number, ω and β are respectively the characteristic velocity and length. We assume the radial component of velocity to be of the form $u = f(\theta)/r$, then continuity eqn. (3.10) shows that $(\partial v/\partial\theta) = 0$. In order to satisfy the no-slip boundary conditions, v has to be zero throughout the field.

Thus

$$u = f(\theta)/r, \quad v = 0. \tag{3.17}$$

Equations (3.15) and (3.16) using (3.17) become

$$R\frac{\partial p}{\partial r} = \frac{f''(\theta)}{r^3} - \frac{M^2 f(\theta)}{r^3} \tag{3.18}$$

$$R\frac{\partial p}{\partial \theta} = \frac{2f'(\theta)}{r^2}. \tag{3.19}$$

Eliminating P between (3.18) and (3.19) we have

$$f''' + 4a^2 f' = 0 \tag{3.20}$$

where

$$a^2 = 1 - \frac{M^2}{4}.$$

We have to solve (3.20) using the boundary conditions (3.14). Equation (3.20) shows that the solutions of eqn. (3.20) are indeterminate at $M = 2$. Thus the cases of $M \neq 2$ and $M = 2$ are treated separately.

Solution for the Case When $M \neq 2$

Solving (3.20) and using separately eqns. (3.17) to (3.19) we have

$$u = \frac{Qa}{r}\,\frac{\sin^2 a\alpha - \sin^2 a\theta}{\sin a\alpha \cos a\alpha - a\alpha + 2a\alpha \sin^2 a\alpha} \tag{3.21}$$

$$v = 0 \tag{3.22}$$

$$R\frac{\partial p}{\partial r} = -\frac{2a}{r^3}\,\frac{\cos 2a\theta}{X} - \frac{aM^2}{r^3}\left(\frac{\sin^2 a\alpha - \sin^2 a\theta}{X}\right) \tag{3.23}$$

$$R\frac{\partial p}{\partial \theta} = \frac{-2a^2 \sin 2a\theta}{Xr^2}\left(a^2 + \frac{M^2}{4}\right) \tag{3.24}$$

$$RP = \frac{a^3 \cos 2a\theta}{r^2 X} + \frac{aM^2}{2r^2}\left(\frac{\sin^2 a\alpha - \sin^2 a\theta}{X}\right)$$

where

$$X = \frac{Q}{\sin a\alpha \cos a\alpha - a\alpha + 2a\alpha \sin^2 a\alpha}.$$

Equations (3.21) to (3.24) satisfy the boundary conditions and differential equations (3.10) to (3.12).

Prior to applying these results to the problem of flow through a channel of varying gap, we transform them to Cartesian system as shown in Fig. 2.

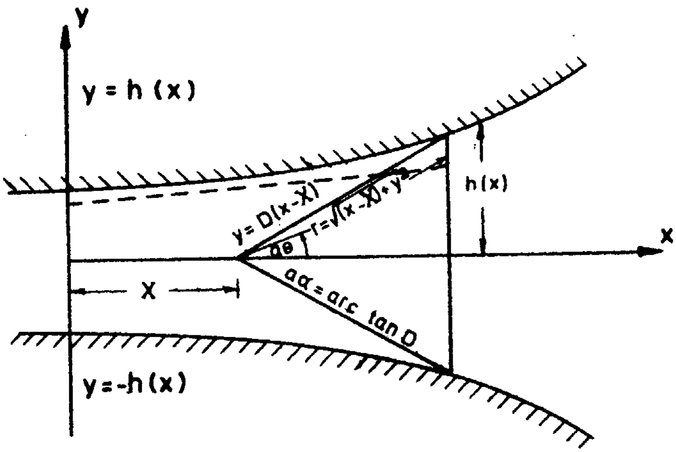

FIG. 2. Flow geometry.

Figure 2, clearly shows

$$u_x = u \cos a\theta, \quad \sin a\theta = \frac{y}{\sqrt{(x-X)^2 + Y^2}},$$

$$u_y = u \sin a\theta, \quad \cos a\theta = \frac{x-X}{\sqrt{(x-X)^2 + Y^2}},$$

$$\sin a\alpha = \frac{D}{\sqrt{1+D^2}}, \quad \cos a\alpha = \frac{1}{\sqrt{1-D^2}},$$

where

$$\tan a\alpha = D, \quad r = \sqrt{(x-X)^2 + y^2}$$

x and y are non-dimensional coordinates. The components of pressure gradients are given by

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \frac{x-X}{r}\frac{\partial p}{\partial r} - \frac{y}{ar^2}\frac{\partial p}{\partial \theta} \\ \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial r}\frac{y}{r} + \frac{x-X}{ar^2}\frac{\partial p}{\partial \theta}. \end{aligned}\right\} \qquad \ldots(3.25)$$

Equations (3.21) to (3.24), using (3.25) become

$$u_x = \frac{Qa(x-X)\, D^2(x-X)^2 - y^2}{E\,\{(x-X)^2 + y^2\}^2} \qquad \ldots(3.26)$$

$$u_y = \frac{QayD^2(x-X)^2 - y^2}{E\{(x-X)^2 + y^2\}^2}, \qquad \text{...(3.27)}$$

$$\frac{\partial p}{\partial x} = \frac{-Qa(x-X)}{RE} \times \frac{[2a^2(1+D^2)\{(x-X)^2 - 3y^2\} + M^2\{D^2(x-X)^2 + y^2 - y^2(1+D^2)\}]}{[(x-X)^2 + y^2]^3}, \qquad \text{...(3.28)}$$

$$\frac{\partial p}{\partial y} = -\frac{Qay}{RE} \times \frac{[2a^2(1+D^2)\{3(x-X)^2 - y^2\} + M^2\{D^2(x-X)^2 - y^2 + (x-X)^2(1+D^2)\}]}{[(x-X)^2 + y^2]^3} \qquad \text{...(3.29)}$$

where

$$E = (\sin a\alpha \cos a\alpha - a\alpha + 2a\alpha \sin^2 a\alpha)(1 + \tan^2 a\alpha)$$

$$= D - (1 - D^2) \text{ arc tan } D.$$

Equations (3.26) to (3.29), using the relation

$$h = D(x - X) \qquad \text{...(3.30)}$$

become

$$u_x = \frac{QaD^3h(h^2 - y^2)}{E(h^2 + D^2y^2)^2} \qquad \text{...(3.31)}$$

$$u_y = \frac{QaD^4y(h^2 - y^2)}{E(h^2 + D^2y^2)^2} \qquad \text{...(3.32)}$$

$$\frac{\partial p}{\partial x} = -\frac{QhaD^3}{RE} \times \frac{[2a^2(1+D^2)(h^2 - 3D^2y^2) + D^2M^2\{(h^2 - y^2) - y^2(1+D^2)\}]}{E(h^2 + D^2y^2)^3} \qquad \text{...(3.33)}$$

$$\frac{\partial p}{\partial y} = -\frac{QayD^4}{RE} \times \frac{[2a^2(1+D^2)\,3h^2 - D^2y^2 + M^2\{D^2(h^2 - y^2) + h^2(1+D^2)\}]}{(h^2 + D^2y^2)^3}. \qquad \text{...(3.34)}$$

If the curvature of wall is everywhere small i.e. $h(x)\,h''(x)$ is small compared to unity then at each value of x the flow in the channel of varying gap may be approximated to a flow in a wedge with vertex at $X(x, 0)$ and vertex angle, 2 arc tan $D(x)$, where

$$\left.\begin{aligned} D(x) &= h'(x), \\ X(x) &= x - \frac{h(x)}{D(x)}. \end{aligned}\right\} \qquad \text{...(3.35)}$$

The velocity components given by (3.31) and (3.32) satisfy the boundary conditions and eqns. (3.33) and (3.34) satisfy the differential equations approximately provided

$$| h(x) h''(x) | \ll 1 \qquad \text{...(3.36)}$$

and

$$| h^2(x) h'''(x) | \ll 1 \qquad \text{...(3.37)}$$

for all x.

If (3.36) and (3.37) is satisfied, then it is very easy to verify that

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \qquad \text{...(3.38)}$$

and we have

$$P = \frac{2QD^3}{R}\left[\int_x^c \frac{(1 + D^2)\, a^3}{Eh^3} dx + \tfrac{1}{2}\int_x^c \frac{D^2 M^2}{Eh^3} dx\right] - \frac{2a^3 QD^4}{RE}$$

$$\times \int_0^y \left[\frac{(1 + D^2)\, 3h^2 y^2 + D^2 y^4 + M^2 \{D^2(h^2 - y^2) + h^2 (1 + D^2)\}}{(h^2 + D^2 y^2)^2}\right] dy \qquad \text{...(3.39)}$$

where C is a constant of integration.

The stream function for the flow is given by

$$\psi(x, y) = \frac{Qa}{2E}\left[\frac{Dh(1 + D^2)\, y}{h^2 + D^2 y^2} - (1 - D^2) \arctan \frac{Dy}{h}\right] \qquad \text{...(3.40)}$$

and hence the velocity components are given by

$$\left.\begin{aligned} u_x &= \frac{\partial \psi}{\partial y} \\ u_y &= -\frac{\partial \psi}{\partial x} + \text{terms in } h(x)\, D'(x). \end{aligned}\right\} \qquad \text{...(3.41)}$$

Third Approximation Method : Power Series Expansion in the Wall Slope

The second method discussed above leads to rather cumbersome results even for analytically simple form of $h(x)$ and also it may happen that the function $h(x)$

satisfying (3.36) and (3.37) is such that $h'(x)$ is small but not negligible. Then the modified condition

$$| D^n | = | h'(x)^n | \ll 1 \qquad ...(3.42)$$

is satisfied for some positive integer n. Now for $n > 1$ the results of the 2nd method are expanded in power series in D and the terms of the nth or higher order in D are neglected. In this method we restrict the analysis to the case when $n = 3$. Expanding arc tan D in power series of D, we have

$$\text{arc tan } D = D - \tfrac{1}{3} D^3 + \tfrac{1}{5} D^5 + ... + O(D^7). \qquad ...(3.43)$$

The function E expanded in terms of D is given by

$$E = \tfrac{4}{3} D^3 [1 - \tfrac{2}{5} D^2 + O(D^4) + ...]. \qquad ...(3.44)$$

Hence

$$\frac{D^3}{E} = \tfrac{3}{4} [1 + \tfrac{2}{5} D^2 + O(D^4) + ...]. \qquad ...(3.45)$$

The expressions for the velocity components and pressure, after neglecting the third and higher order in D, are

$$u_x = \frac{3Qa}{4h} [1 - (y/h)^2] [1 - 2D^2(y/h)^2 + \tfrac{2}{5} D^2 + ...] \qquad ...(3.46)$$

$$u_y = \frac{3QaD}{4h} \left(\frac{y}{h}\right) [1 - (y/h)^2] \qquad ...(3.47)$$

$$\frac{\partial p}{\partial x} = \frac{-3Qa}{Rh^3} [a^2 - 6a^2 D^2 (y/h)^2 + \tfrac{7}{5} a^2 D^2 + \tfrac{1}{2} D^2 M^2 (1 - 2y^2/h^2)] \qquad ...(3.48)$$

$$\frac{\partial p}{\partial y} = - \frac{QaD}{Rh^3} \frac{y}{h} \left[\frac{18a^2}{4} + \tfrac{3}{4} M^2\right] \qquad ...(3.49)$$

$$\psi(x, y) = \frac{3Qya}{4h} \left\{1 - \tfrac{1}{3}\left(\frac{y}{h}\right)^2 + \tfrac{2}{5} D^2 \left(1 - \left(\frac{y}{h}\right)^2\right)\right\} \qquad ...(3.50)$$

$$P = \frac{3Q}{4R} \left[2a^3 \int_x^c \frac{1 + \tfrac{7}{5} D^2}{h^3} dx + \int_x^c \frac{M^2}{2h^3} dx - \left\{\frac{3Dy^2a^3 + M^2y^2}{h^4}\right\}\right]. \qquad ...(3.51)$$

Solutions for the Case When $M = 2$

The above analysis is restricted to the case when $M \neq 0$, for the solutions (3.21) to (3.24) become indeterminate when $M = 2$. However for $M = 2$, solutions can be obtained by appealing to eqns. (3.18) to (3.20) and the solutions are of the form

$$u = \frac{3Q}{4\alpha^3 r}(\alpha^2 - \theta^2) \qquad ...(3.52)$$

$$R\frac{\partial p}{\partial r} = \frac{3Q}{2\alpha^2 r^2}(2\theta^2 - 2\alpha^2 - 1) \qquad ...(3.53)$$

$$R\frac{\partial p}{\partial \theta} = -\frac{3Q}{4\alpha^3 r^2} \qquad ...(3.54)$$

$$RP = -\frac{3Q}{4\alpha^3 r^2}(2\theta^2 - 2\alpha^2 - 1). \qquad ...(3.55)$$

Expressing θ and α in terms of D

$$\theta = \text{arc tan}\,\frac{Dy}{h} \quad \text{and} \quad \alpha = \text{arc tan}\, D$$

we have

$$u_x = \frac{3QDh}{4\,(\text{arc tan}\, D)^3}\left[\frac{(\text{arc tan}\, D)^2 - (\text{arc tan}\,(Dy/h))^2}{h^2 + D^2y^2}\right] \qquad ...(3.56)$$

$$u_y = \frac{3QD^2y}{4\,(\text{arc tan}\, D)^3}\left[\frac{(\text{arc tan}\, D)^2 - (\text{arc tan}\,(Dy/h))^2}{h^2 + D^2y^2}\right] \qquad ...(3.57)$$

$$\frac{\partial p}{\partial y} = \frac{3QD^4}{R\,(\text{arc tan}\, D)^3} \times \left[\frac{y\,\{(\text{arc tan}\,(Dy/h))^2 - (\text{arc tan}\, D)^2\} - (h/D)\,\text{arc tan}\,(Dy/h)}{(h^2 + D^2y^2)^2}\right] \qquad ...(3.58)$$

$$\frac{\partial p}{\partial x} = \frac{3QD^3}{R\,(\text{arc tan}\, D)^3} \times \frac{[h\,\{(\text{arc tan}\,(Dy/h))^2 - (\text{arc tan}\, D)^2 - \tfrac{1}{2}\} - yD\,\text{arc tan}\,(Dy/h)]}{(h^2 + D^2y^2)^2} \qquad ...(3.59)$$

$$\psi = \frac{3Q}{4}\left[\frac{\text{arc tan}\,(Dy/h)}{\text{arc tan}\, D} - \frac{(\text{arc tan}\,(Dy/h))^3}{3\,(\text{arc tan}\, D)^3}\right]. \qquad ...(3.60)$$

If the above results are expanded in power series of D, the wall slope, then the expressions for the velocity components and pressure gradients are given by

$$u_x = \frac{3Q}{4h}\left\{1 - \frac{y^2}{h^2}\right\}\left\{1 - \frac{D^2y^2}{h^2}\right\} \qquad ...(3.61)$$

$$u_y = \frac{3QD}{4h}\left(\frac{y}{h}\right)\left(1 - \frac{y^2}{h^2}\right) \qquad ...(3.62)$$

$$\frac{\partial p}{\partial x} = -\frac{3Q}{Rh^3}\left\{1 - \frac{2D^2y^2}{h^2} + D^2\right\} \qquad ...(3.63)$$

$$\frac{\partial p}{\partial y} = -\frac{3QD}{Rh^3}(y/h) \qquad \ldots(3.64)$$

$$\psi = \frac{3Q}{4}(y/h)\left[1 + D^2\left\{\frac{y}{3h} - \frac{2}{3}\frac{y^2}{h^2}\right\}\right]. \qquad \ldots(3.65)$$

Comparison of Three Methods

The average pressure gradient across the channel is

$$\Delta x = -\frac{1}{h(x)}\int_0^{h(x)} \frac{\partial p}{\partial x}\,dy. \qquad \ldots(3.66)$$

Δx is calculated in the neighbourhood of a given value of x with each of the methods and the weight functions are compared. Δx is found to be

$$\Delta x = \frac{3Qa}{2Rh^3}F_i(D), \text{ when } i = 1, 2, 3, \text{ when } M \neq 2, \qquad \ldots(2.67)$$

where $F_i(D)$, $i = 1, 2, 3$ depends upon the method used. For $M \neq 2$

$$F_1(D) = \left[\frac{M^3 \cosh M}{3a\,(M \cosh M - \sinh M)}\right], \qquad \ldots(3.68)$$

$$F_2(D) = \frac{1}{E}\left[\frac{4a^2D^3}{3(1 + D^2)} + \frac{M^2D^2}{6(1 + D^2)}\{D^3 + D + (D^4 - 1)\tan^{-1} D\}\right] \qquad \ldots(3.69)$$

$$F_3(D) = \left[a^2(1 - \tfrac{3}{5}D^2) + \frac{D^2M^2}{6}\right]. \qquad \ldots(3.70)$$

Three weight functions are numerically evaluated for values of M equal to 0 and 1 and are represented Figs. 3 and 4.

4. Flow Through a Channel with a Smooth Constriction

The above theory is applied to the problem of flow through a channel with smooth, axi-symmetric constriction (Fig. 5) defined in non-dimensional variables, by

$$h(x) = H_0 - \frac{\delta_m}{2}\left(1 + \cos\frac{\pi x}{x_0}\right) \qquad \ldots(4.1)$$

where δ_m is the maximum projection of the constriction and H_0 is the half width of the channel.

The third approximation method is valid, when the condition (3.42) is satisfied, viz,

$$|\,D^n\,| = |\,h'(x)^n\,| \ll 1$$

for some positive integer n.

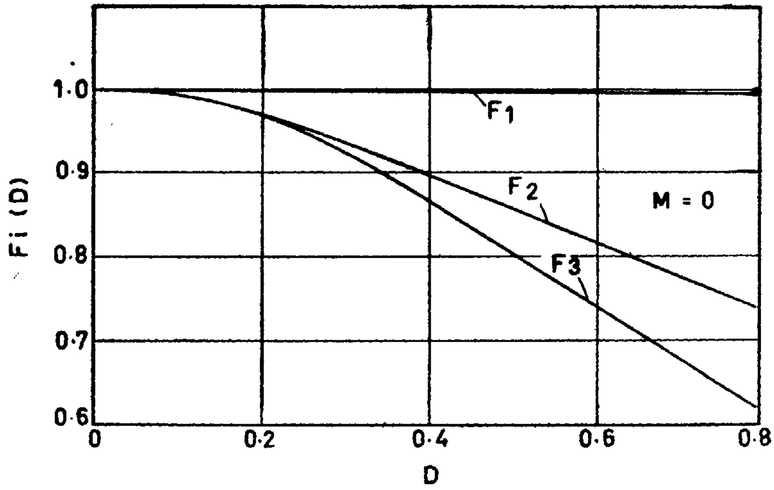

FIG. 3. Comparison of weight functions.

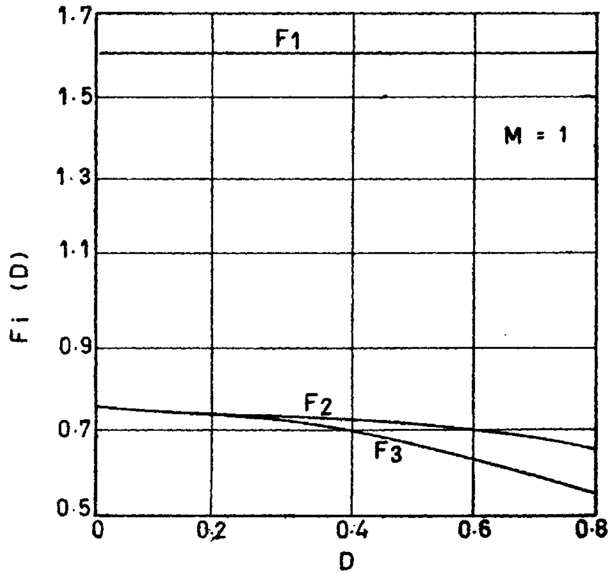

FIG. 4. Comparison of weight functions.

The expression for D, from (4.1), is

$$D = h'(x) = \frac{\pi\delta_m}{2x_0} \sin \frac{\pi x}{x_0}. \qquad ...(4.2)$$

We note that the condition $| D^n | \ll 1$ will be satisfied if the following non-dimensional quantities take the values

$$L = 4.0$$

$$x_0 = 1.0$$

$$\delta_m = 0.32 H_0$$

$$H_0 = 1.0.$$

For these values $|D|$ has a maximum value of 0.5 at $x = x_0/2$ and satisfies the condition (3.42) for all positive values of n.

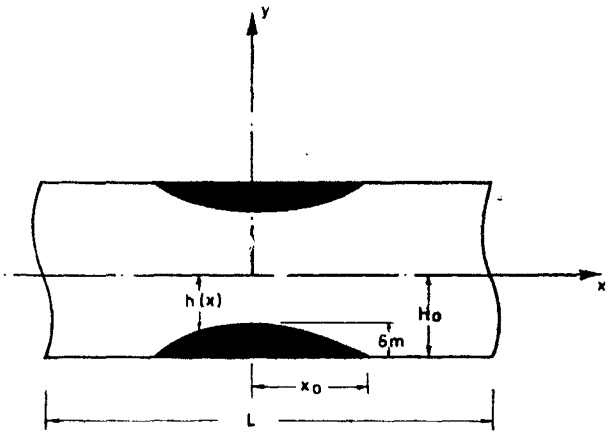

FIG. 5. Idealized geometry for stenosis.

To determine the effect of the magnetic field on the flow characteristics of the stenosis, it is essential to determine the resistance to flow and the shear stress at the wall. The resistance to flow denoted by R.F is defined as :

$$\text{R.F.} = \frac{\text{average pressure drop across the channel}}{\text{flux in the direction of flow}}. \quad \text{...(4.3)}$$

To determine R.F. the average pressure drop across the channel and the flux in the direction of flow are to be known. The non-dimensional average pressure drop across the channel is calculated from the expression

$$P_0 - P = \frac{1}{2hL} \int_{-h(x)}^{+h(x)} \int_0^L \frac{\partial P}{\partial x}\, dx\, dy. \quad \text{...(4.4)}$$

Using (3.48) and integrating over the interval $-h$ to $+h$ (4.4) takes the form

$$P = \int_0^L \frac{3a}{2hL} \left[\frac{2a^2(1 - \frac{3}{5} D^2) + \frac{1}{3} D^2 M^2}{h^2} \right] dx. \quad \text{...(4.5)}$$

Equation (4.5) can be written in the form

$$P = \frac{3a^2}{hL}(I_1 + I_2) \qquad \text{...(4.6)}$$

where

$$I_1 = \int_0^{x_0/2} \frac{[2a(1 - \frac{3}{5}D^2) + \frac{1}{3}D^2M^2]}{h^2}\, dx$$

$$I_2 = \int_{x_0/2}^{L/2} \frac{2a}{H_0^2}\, dx.$$

The non-dimensional momentum flux in the horizontal direction has the form

$$M = \int_{-h}^{h} u_x^2\, dy \qquad \text{...(4.7)}$$

and after performing the indicated integration, simplifies to,

$$M = \frac{9}{4}\,\frac{Q^2a^2K}{h} \qquad \text{...(4.8)}$$

where

$$\left.\begin{aligned}
K &= A_1 + \frac{A_2}{3} + \frac{A_3}{5} + \frac{A_4}{7} + \frac{A_5}{9}\\
A_1 &= \left(1 + \frac{2}{5}D^2\right)^2\\
A_2 &= -2\left(1 + \frac{2}{5}D^2\right)\left(1 + \frac{12}{5}D^2\right)\\
A_3 &= \left(1 + \frac{12}{5}D^2\right)^2 + 4D^2\left(1 + \frac{2}{5}D^2\right)\\
A_4 &= -\,4D^2\left(1 + \frac{12}{5}D^2\right)\\
A_5 &= 4D^4.
\end{aligned}\right\} \qquad \text{...(4.9)}$$

The expression for Resistance to Flow, using (4.6) and (4.8), now becomes

$$\text{R.F.} = \frac{4}{3}\,\frac{(I_1 + I_2)}{Q^2KL}. \qquad \text{...(4.10)}$$

Analytical evaluation of the integral I_1 is complicated and hence it is evaluated numerically on a computer. The R.F. was calculated for different values of Hartmann number, M, and the behaviour of R.F. with M is presented in Fig. 6. From this it is clear that the R.F. decreases with the increase in the values of M and the decrease

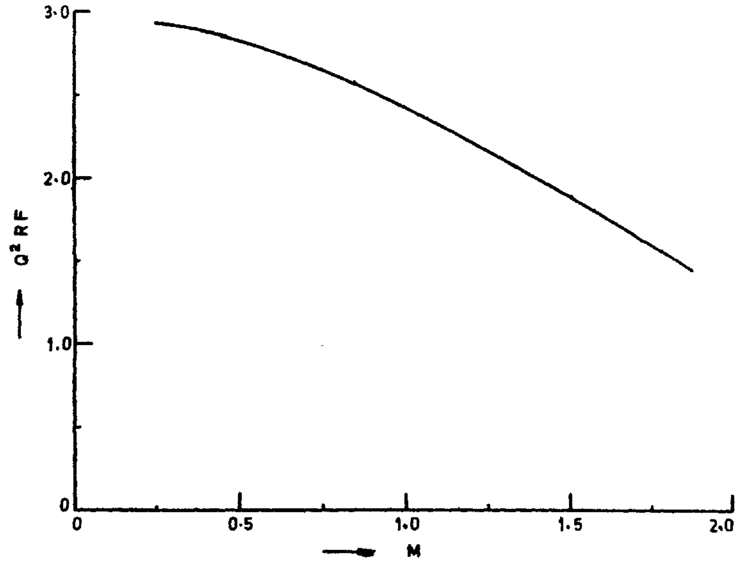

FIG. 6. Resistance force versus Hartmann number

is significant (nearly 50 per cent for $M = 1.75$). This is in conformity with our discussion in section 1.

The resistance given by (4.10) is computed using momentum flux. However in bio-mechanical problems, for example, in the study of blood rheology in arterial flows, the resistance to flow is normally defined (Lightfoot 1974) as

$$\text{R.F.} = \frac{\text{pressure drop}}{\text{volumetric flow rate}} = \frac{3a(I_1 + I_2)}{Qhl} \quad \text{...(4.11)}$$

The resistance to flow given by (4.11) is numerically computed for different values of M and the results are compared with those of (4.10). We observe that the overall nature of the resistance to flow with the magnetic field is the same, whether we use (4.10) or (4.11) except for a slight change in the magnitude.

The non-dimensional shear stress at the wall is next calculated using the expression

$$(\tau_{xy})_{y=h} = \frac{1}{R}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)_{y=h}$$

$$= \frac{3ad}{2h^2R}(1 - 3.1D^2) \quad \text{...(4.12)}$$

for different values of M and the results are presented in Tables I and II. It is seen that the shear stress also exibits a decreasing tendency with increase in M but the decrease is not as significant as R.F.

TABLE I

$D = 0,\ x = 0$

M	a	h	$\tau = \frac{3a}{2h^2}$
0.5	0.9375	0.68	3.0412
0.75	0.9270	0.68	3.0070
1.0	0.8660	0.68	2.8092
1.5	0.6614	0.68	2.1455

TABLE II

$D = 0.5,\ x = \frac{x_0}{2}$

M	a	h	$\tau = \frac{3a}{2h^2}(1 - 3.1D^2)$
0.5	0.9375	0.84	0.44842
0.75	0.9270	0.84	0.44349
1.0	0.8660	0.84	0.41220
1.5	0.6614	0.84	0.31560

5. CONCLUSIONS

Three approximate methods have been presented for solving the problem of flow through a channel of varying gap in the presence of a magnetic field. These methods provide an alternate approach to the conventional method of solving a two dimensional problem by conformal mapping. The weight functions are calculated with each of the methods for two values of $M (M = 0$ and $1)$ and are shown in Figs. 3 and 4. It is seen that the weight functions $F_2(D)$ and $F_3(D)$ are influenced by the magnetic field and have values smaller than those in the absence of a magnetic field. To study the impact of magnetic field on stenosis an idealized stenosis geometry is considered and the flow characteristics such as resistance to flow and shear stress are determined. The overall effect of magnetic field as seen from Fig. 6 and Tables I and II is to decrease the resistance to flow and shear stress at the wall and thus reduce the abnormalities due to irregular boundaries.

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