Indian J. pure appl. Math., 11(5) : 547-561, May 1980

ON CONVERGENCE OF DERIVATIVES OF POST-WIDDER OPERATORS

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(*Received* 10 *July* 1979)

Simultaneous approximation property of the Post-Widder operators has been established. An asymptotic formula in the simultaneous approximation has been obtained. Saturation and inverse theorems in the simultaneous approximation have been deduced.

1. INTRODUCTION

Let $\mathbf{N}$, $\mathbf{R}^+$ and $\bar{f}$, respectively, denote the set of natural numbers, the interval $(0, \infty)$ and the Laplace transform of a function f. Let p be a fixed integer. The nth Post-Widder operator $L_{n,x}$ is defined by

$$L_{n,x}\bar{f} = \frac{(-1)^{n+p}}{(n+p)!}\,\bar{f}^{(n+p)}\left(\frac{n}{x}\right)\left(\frac{n}{x}\right)^{n+p+1}, \; x \in \mathbf{R}^+ \qquad \text{...(1.1)}$$

and it exists for all sufficiently large $n \in \mathbf{N}$, if $\bar{f}(t)$ exists for some $t \in \mathbf{R}^+$. Following Widder (1946, p. 288) an integral representation for $L_{n,x}$ is as follows:

$$L_{n,x}\bar{f} = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1} \int_0^\infty f(t)\, t^{n+p} e^{-nt/x}\, dt. \qquad \text{...(1.2)}$$

The inversion formula

$$\lim_{n\to\infty} L_{n,x}\bar{f} = f(x) \qquad \text{...(1.3)}$$

which holds at each continuity point x of f, was investigated by Widder (1946) for $p = 0$ and by May (1976) for $p = -1$. Using the representation (1.2), the result for a general p can be established analogously.

Let $\{L_n\}$, $n \in \mathbf{N}$, be a sequence of operators defined on a domain D of functions. We say that $\{L_n\}$ has simultaneous approximation property of order $k(\in \mathbf{N})$ if for an arbitrary $f \in D$, the kth derivative of $(L_n f)(x)$ converges to the kth derivative of $f(x)$ whenever the latter exists.

Lorentz (1953) has studied simultaneous approximation properties of Bernstein polynomials. Lupas and Müller (1970) have established the simultaneous approximation property of M_n-operators of Meyer-König and Zeller. Rathore (1978) has obtained an asymptotic formula in the simultaneous approximation of M_n-operators. Martini (1969) discusses simultaneous approximation property of Baskakov operators.

In this paper, we establish the simultaneous approximation property of the operators $L_{n,x}$, namely the convergence $L_{n,x}^{(k)} \bar{f} \to f^{(k)}(x)$ as $n \to \infty$ ($k \in \mathbb{N}$, $x \in \mathbb{R}^+$), whenever the Laplace transform $\bar{f}$ of f and $f^{(k)}(x)$ exist, and obtain an asymptotic formula in the simultaneous approximation. We also discuss direct, inverse and saturation theorems for $L_{n,x}^{(k)} \bar{f}$ and their linear combinations. Throughout the paper $\bar{f}$ stands for Laplace transform of f and f's are such that $\bar{f}$'s exist at some point $t \in \mathbb{R}^+$.

2. Auxiliary Results

Lemma 1 — There exist polynomials $q_{ijk}(x)$ in x which do not depend on t or n such that

$$\frac{\partial^k}{\partial x^k} (x^{-(n+p+1)} e^{-nt/x}) = Q_k(x)\, x^{-n-p-1-2k} e^{-nt/x} \qquad \ldots(2.1)$$

where

$$Q_k(x) = \sum_{i,j} n^{i+j} (t - x)^j q_{ijk}(x),\ i, j \geqslant 0,\ 2i + j \leqslant k. \qquad \ldots(2.2)$$

Proof : Following Lorentz (1953) and using an induction on k, the lemma can be easily established.

Let $0 < \delta < x$, $I_\delta(x) = (x - \delta, x + \delta)$ and $J_\delta(x) = \mathbb{R}^+ \setminus I_\delta(x)$.

Lemma 2 — If f is such that $\bar{f}(t)$ exists for some $t \in \mathbb{R}^+$ then for an arbitrary $s > 0$,

$$\frac{1}{(n+p)!} \left(\frac{n}{x}\right)^{n+p+1} \int_{J_\delta(x)} f(t)\, t^{n+p} e^{-nt/x}\, dt = o(n^{-s}),\ n \to \infty. \qquad \ldots(2.3)$$

Also, if $0 < \delta < a < b < \infty$, then relation (2.3) holds uniformly in $x \in [a, b]$.

Proof : For two arbitrary numbers

$\delta_0, A_0 > 0,\ 0 < \delta_0 < x - \delta < x + \delta < A_0 < \infty$ we have

$$\left| \frac{1}{(n+p)!} \left(\frac{n}{x}\right)^{n+p+1} \int_{[\delta_0, A_0] \setminus I_\delta(x)} f(t)\, t^{n+p} e^{-nt/x}\, dt \right|$$

$$\leqslant \frac{n^{n+p+1}}{(n+p)!} \int_{[\delta_0, A_0] \setminus I_\delta(x)} \left| f(t) \frac{t^p}{x^{p+1}} \right| \left(\frac{x+\delta}{x} e^{-(x+\delta)/x} \right)^n dt$$

$$\cong C\sqrt{n/2\pi}\left(\frac{x+\delta}{x}e^{-\delta/x}\right)^n \qquad \text{(by Stirling's formula)}$$

where $$C = \int_{[\delta_0, A_0]\setminus I_\delta(x)} \left| f(t)\frac{t^p}{x^{p+1}} \right| dt.$$

It follows that

$$\frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1} \int_{[\delta_0, A_0]\setminus I_\delta(x)} f(t)\, t^{n+p} e^{-nt/x}\, dt = o(n^{-s}), \quad n \to \infty \qquad \ldots(2.4)$$

for an arbitrary $s > 0$. Further, (2.4) holds uniformly in $x \in [a, b]$ provided δ_0, A_0 are fixed and $0 < \delta_0 < a < b < A_0 < \infty$. Now, since $\bar{f}$ exists, there exist $M, b' > 0$ such that

$$\int_0^t | f(x) | \, dx < M, \quad 0 < t < b'.$$

Let us choose $\delta_0 > 0$ such that for some $1 > \epsilon > 0$, $\delta_0 < \min\left\{b', \frac{x(1-\epsilon)}{e}\right\}$ $\left(\text{for } x \in [a, b], \text{ we choose } \delta_0 < \min\left\{b', \frac{a(1-\epsilon)}{e}\right\}\right)$. Then it is easy to see that

$$\frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1} \int_0^{\delta_0} f(t)\, t^{n+p} e^{-nt/x}\, dt = o(n^{-s}), \; n \to \infty \qquad \ldots(2.5)$$

for each $s > 0$ (and uniformly for $x \in [a, b]$ in the uniformity case). Further, since $\bar{f}$ exists, there exist p, A_p, $a_p > 0$ such that

$$\int_t^\infty | f(x) | \, e^{-px} \, dx < A_p, \quad t > a_p.$$

Now, it is clear that if we choose a number $q_0 > 0$ such that for some $\epsilon > 0$, $q_0 < (1/x) - \epsilon$ (in the uniformity case $q_0 < (1/b) - \epsilon$) then there exists a $t_0 > 0$ such that for all $t > t_0$, $t < e^{q_0 t}$. If we choose $A_0 > \max\left\{a_p, t_0, \frac{1}{\epsilon}\log(e/x)\right\}$ $\left(\text{in the uniformity case } A_0 > \max\left\{a_p, t_0, \frac{1}{\epsilon}\log(e/a)\right\}\right)$, then it is easy to verify that

$$\frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1} \int_{A_0}^\infty f(t)\, t^{n+p} e^{-nt/x}\, dt = o(n^{-s}), \quad n \to \infty \qquad \ldots(2.6)$$

for each $s > 0$, and that (2.6) holds uniformly in $x \in [a, b]$ in the uniformity case. Equations (2.4) – (2.6) establish (2.3). This completes the proof of the lemma.

If $f(t) = t^i$, then using the definition (1.2) one easily obtains

$$L_{n,x}\bar{f} = \frac{\Gamma(n + i + p + 1)}{\Gamma(n + p + 1)\, n^i}\, x^i \qquad ...(2.7)$$

where i is a number satisfying $n > -(i + 1)$. The rth moment

$$\mu_{n,r}(x) \ (r = 0, 1, 2, ...)$$

of the operator $L_{n,x}$ is defined by

$$\mu_{n,r}(x) = L_{n,x}\bar{g},$$

where $g(t) = (t - x)^r$. There holds the following recurrence formula for the moments $\mu_{n,k}(x)$:

$$\mu_{n,k+1}(x) = \frac{x}{n} \{(p + k + 1)\, \mu_{n,k}(x) + kx\mu_{n,k-1}(x)\} \ (k = 1, 2, ...). \qquad ...(2.8)$$

Lemma 3 — For $k = 0, 1, 2, ..., x^{-k}\mu_{n,k}(x)$ does not depend on x and

$$\sigma_{n,k} = O(n^{-[(k+1)/2]}) \qquad ...(2.9)$$

where $\sigma_{n,k} = x^{-k}\mu_{n,k}(x)$ and $\left[\frac{k+1}{2}\right]$ denotes the integral part of $\frac{1}{2}(k + 1)$.

PROOF : The lemma can be easily proved by using the recurrence formula (2.8) and an induction on k.

An application of Lemma 3 and Schwarz's inequality gives

$$L_{n,x}\bar{f} = O(n^{-k/2}) \qquad ...(2.10)$$

where $f(t) = |t - x|^k$.

Theorem 1 — If f is such that $\bar{f}$ and $f^{(k)}(x)$ exist for some point $x \in \mathbf{R}^+$, then

$$L_{n,x}\bar{f} = \sum_{r=0}^{k} \frac{f^{(r)}(x)}{r!}\, \mu_{n,r}(x) + o(n^{-k/2}),\ n \to \infty. \qquad ...(2.11)$$

Further, if $f^{(k)}$ exists in $\langle a, b \rangle$ (an open interval containing $[a, b]$) $\subset \mathbf{R}^+$ and is continuous at each $x \in [a, b]$, then (2.11) holds uniformly in $x \in [a, b]$.

PROOF : The theorem follows from Taylor's expansion of f, eqn. (2.10) and Lemma 2.

Definition 1 — The mth linear combination of $L_{n,x}\bar{f}$ denoted by $L^{[m]}_{n,x}\bar{f}$ is defined by

$$L_{n,x}^{[m]}\bar{f} = \frac{1}{\Delta m}\begin{vmatrix} L_{\alpha_0 n,x}\bar{f} & 1/\alpha_0 & 1/\alpha_0^2 & \dots & 1/\alpha_0^{m-1} \\ L_{\alpha_1 n,x}\bar{f} & 1/\alpha_1 & 1/\alpha_1^2 & \dots & 1/\alpha_1^{m-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ L_{\alpha_{m-1} n,x}\bar{f} & 1/\alpha_{m-1} & 1/\alpha_{m-1}^2 & \dots & 1/\alpha_{m-1}^{m-1} \end{vmatrix} \qquad \dots(2.12)$$

where α_i's are certain fixed positive real numbers and Δ_m is the determinant obtained by replacing the elements in the first column of the above determinant by the entries 1. Linear combinations of this type were introduced by Rathore (1973).

Linear combinations of $L_{n,x}^{(k)}\bar{f}$ are denoted by $L_{n,x}^{(k)[m]}\bar{f}$. Obviously there holds

$$L_{n,x}^{[m](k)}\bar{f} = L_{n,x}^{(k)[m]}\bar{f}.$$

Definition 2 — Let $\Delta_x^k f(t) = \sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu} f(t+\nu x)$ and let

$$\omega_k(f, h; a, b) = \sup \{ | \Delta_x^k f(t) | ; | x | \leqslant h; t, t + kx \in [a, b]\}$$

be the kth modulus of continuity of the function f. Then the class Liz $(\alpha, k; a, b)$ is the class of functions for which $\omega_{2k}(f, h; a, b) \leqslant Mh^{\alpha}$. When $k = 1$, Liz $(\alpha, 1)$ reduces to the well-known Zygmund class Z_α ($\equiv$ Lip* α).

Definition 3 — The operators $S_n^1(f; x)$ which have also been termed Post-Widder operators by May (1976) are defined by

$$S_n^1(f; x) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n \int_0^\infty f(t)\, t^{n-1} e^{-nt/x}\, dt. \qquad \dots(2.13)$$

Obviously, $L_{n,x}\bar{f}$ reduces to $S_n^1(f; x)$ on taking $p = -1$.

Also,

$$L_{n,x}\bar{f} = \begin{cases} \dfrac{1}{(n+p).(n+p-1)\dots n}\left(\dfrac{n}{x}\right)^{p+1} S_n^1(t^{p+1}f(t); x), \text{ if } p > -1 \\ (n-1).(n-2)\dots(n+p+1)\left(\dfrac{n}{x}\right)^{p+1} S_n^1(t^{p+1}f(t); x), \text{ if } p < -1 \end{cases} \qquad \dots(2.14)$$

Theorem 2 (*Inverse theorem*) — Let f be such that $\bar{f}(t)$ exists for some $t \in \mathbf{R}^+$, $0 < \alpha < 2$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then, in the following the implications (1) ⇒ (2) ⇔ (3) ⇒ (4) hold:

(1) $\sup_{x \in [a_1, b_1]} | L_{n_r,x}^{[m+1]} \bar{f} - f(x) | = O(n_r^{-\alpha(m+1)/2})$

$$\left(\frac{n_{r+1}}{n_r} \leqslant c \text{ for some constant } c > 0\right);$$

(2) $f(x) \in \mathrm{Liz}\,(\alpha, m+1; a_2, b_2)$;

(3) (i) For $m' < \alpha(m+1) < m'+1$, $m' = 0, 1, 2, \ldots, 2m+1$, $f^{(m')}$ exists and belongs to $\mathrm{Lip}\,(\alpha(m+1) - m'); a_2, b_2)$,

(ii) For $\alpha(m+1) = m'+1$, $m' = 0, 1, 2, \ldots, 2m$, $f^{(m')}(x)$ exists and belongs to $Z_1(a_2, b_2)$;

(4) $\| L_{n,x}^{[m+1]} \bar{f} - f \|_{C[a_3, b_3]} = O(n^{-\alpha(m+1)/2})$.

Theorem 3 (*Saturation theorem*) — Let f be such that $\bar{f}(t)$ exists for some $t \in \mathbf{R}^+$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then, in the following the implications (1) ⇒ (2) ⇒ (3) and (4) ⇒ (5) ⇒ (6) hold:

(1) $n_r^{m+1} \sup_{x \in [a_1, b_1]} | L_{n_r,x}^{[m+1]} \bar{f} - f(x) | = O(1) \left(\frac{n_{r+1}}{n_r} \leqslant c\right)$;

(2) $f^{(2m+1)} \in A.C.\,[a_2, b_2]$ and $f^{(2m+2)} \in L_\infty\,[a_2, b_2]$;

(3) $n^{m+1} \| L_{n,x}^{[m+1]} \bar{f} - f \|_{C[a_3, b_3]} = O(1)$;

(4) $n_r^{m+1} \sup_{x \in [a_1, b_1]} | L_{n_r,x}^{[m+1]} \bar{f} - f(x) | = o(1) \left(\frac{n_{r+1}}{n_r} \leqslant c\right)$;

(5) $f \in C^{2m+2}\,[a_2, b_2]$ and $\sum_{i=m}^{2m+2} Q(i, m, x)\, f^{(i)}(x) = 0$, $x \in [a_2, b_2]$,

where $Q(i, m, x)$ are certain polynomials depending on m;

(6) $n^{m+1} \| L_{n,x}^{[m+1]} \bar{f} - f \|_{C[a_3, b_3]} = o(1)$.

When $f \in C(\mathbf{R}^+)$, Theorems 2–3 follow from results of May (1976), (2.14) and the fact that $f(x) \in \mathrm{Liz}\,(\alpha, m+1; a, b)\,(a > 0)$ if, and only if $xf(x) \in \mathrm{Liz}\,(\alpha, m+1; a, b)$. For a general f, the theorems follow from an observation made in section 5.

3. Simultaneous Approximation Property and Asymptotic Formulae in the Simultaneous Approximation

The following theorem establishes the simultaneous approximation property of the operators $L_{n,x}$:

Theorem 4 — Let f be such that $\bar{f}$ and $f^{(k)}(x)$ exist for some $x \in \mathbf{R}^+$, then

$$\lim_{n\to\infty} L^{(k)}_{n,x}\,\bar{f} = f^{(k)}(x). \qquad ...(3.1)$$

Further, if $f^{(k)}(x)$ exists in $\langle\, a, b\,\rangle \subset \mathbf{R}^+$ and is continuous at each $x \in [a, b]$, then (3.1) holds uniformly for $x \in [a, b]$.

PROOF: If $\bar{f}$ and $f^{(k)}(x)$ exist for some $x \in \mathbf{R}^+$, then

$$f(t) = \sum_{r=0}^{k} \frac{f^{(r)}(x)}{r!}(t-x)^r + h(t;x) \qquad ...(3.2)$$

where $h(t;x)$ is such that $\bar{h}(t;x)$ exists and given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\,h(t;x)\,| \leqslant \epsilon\,|\,t-x\,|^k,\ |\,t-x\,| < \delta. \qquad ...(3.3)$$

Further, if $f^{(k)}(x)$ exists in $\langle\, a, b\,\rangle$ and is continuous at each $x \in [a, b]$, the above δ can be chosen to be independent of x with the inequality holding for all $x \in [a, b]$. Operating eqn. (3.2) by $L^{(k)}_{n,x}$, we have

$$L^{(k)}_{n,x}\,\bar{f} = \left\{\frac{\partial^k}{\partial y^k} L_{n,y}\,\bar{g}\right\}_{y=x} + L^{(k)}_{n,x}\,\bar{h}$$

$$= \sum_{r=0}^{k} \frac{f^{(r)}(x)}{r!} \left\{\frac{\partial^k}{\partial y^k} \sum_{i=0}^{r} \binom{r}{i} (-x)^{r-i} y^i \times \frac{\Gamma(n+p+i+1)}{\Gamma(n+p+1)\,n^i}\right\}_{y=x} + L^{(k)}_{n,x}\,\bar{h},$$

from (2.7), where $g(t) = \sum_{r=0}^{k} \frac{f^{(r)}(x)}{r!}(t-x)^r$.

Since the first term in the last expression simplifies to

$$\frac{\Gamma(n+p+k+1)}{\Gamma(n+p+1)\,n^k}\, f^{(k)}(x),$$

in order to prove (3.1) it is sufficient to show that

$$\lim_{n\to\infty} L^{(k)}_{n,x}\,\bar{h} = 0 \qquad ...(3.4)$$

and that (3.4) holds uniformly in the uniformity case. By Lemma 1, we have

$$L^{(k)}_{n,x}\,\overline{h\chi_{I_\delta(x)}} = x^{-2k} L_{n,x}\bar{G},$$

where $G(t) = h(t; x)\, \chi_{I_\delta(x)}(t) \sum_{i,j \geqslant 0} n^{i+j}(t-x)^j q_{ijk}(x)$ and $\chi_{I_\delta(x)}$ denotes the characteristic function of the interval $I_\delta(x)$. Now fix a pair (i, j). Then, by the positivity of $L_{n,x}$ and eqn. (2.10), we have

$$| L_{n,x} g_j | \leqslant \epsilon M_{j+k} n^{-(j+k)/2}\, x^{j+k}$$

where $g_j(t) = h(t; x)\,(t-x)^j \chi_{I_\delta(x)}(t)$ and M_{j+k} is a constant depending on j and k. Hence

$$| x^{-2k} q_{ijk}(x)\, n^{i+j} L_{n,x} g_j | \leqslant \epsilon M_{j+k}\, | q_{ijk}(x) |\, x^{j-k} n^{(2i+j-k)/2}.$$

Since $2i + j \leqslant k$, it follows that there exists a constant $C(x, k)$, not depending on n or ϵ, such that

$$| L^{(k)}_{n,x} \overline{h\chi}_{I_\delta(x)} | < \epsilon C(x, k) \qquad \text{for all } n \geqslant 1. \qquad ...(3.5)$$

Now, with $J_\delta(x) = \mathbf{R}^+ \setminus I_\delta(x)$ and $g_j^*(t) = h(t; x)\, \chi_{J_\delta(x)}(t)\,(t-x)^j$, by Lemma 1, we have

$$L^{(k)}_{n,x} \overline{h\chi}_{J_\delta(x)} = \sum_{\substack{i,j \geqslant 0 \\ 2i+j \leqslant k}} n^{i+j} q_{ijk}(x)\, x^{-2k} L_{n,x} \bar{g}_j^*$$

which, by Lemma 2, is seen to be of $o(n^{-s})$, $n \to \infty$, for each $s > 0$.

Hence

$$\lim_{n \to \infty} L^{(k)}_{n,x} \overline{h\chi}_{J_\delta(x)} = 0. \qquad ...(3.6)$$

From (3.5) - (3.6), due to the arbitrariness of ϵ, we have (3.4). Further, (3.4) holds uniformly in the uniformity case. This completes the proof of Theorem 4.

The following theorem gives an asymptotic formula of the type obtained by Rathore (1978) in the simultaneous approximation of M_n-operators.

Theorem 5 — Let f be such that $\bar{f}$ and $f^{(k+2)}(x)$ exist for some $x \in \mathbf{R}^+$, then

$$L^{(k)}_{n,x} \bar{f} - f^{(k)}(x) = \frac{1}{2n} [f^{(k)}(x)\, k(2p+k+1) + 2f^{(k+1)}(x)(p+k+1)x + f^{(k+2)}(x)\, x^2] + o\left(\frac{1}{n}\right), \quad n \to \infty. \qquad ...(3.7)$$

Further, if $f^{(k+2)}(x)$ exists in $\langle a, b \rangle \subset \mathbf{R}^+$ and is continuous at each $x \in [a, b]$, then (3.7) holds uniformly in $x \in [a, b]$.

PROOF: Taking $m = 1$ in the subsequent Theorem 6, Theorem 5 follows after simple computations.

In the following theorem we obtain a generalized asymptotic formula which is used in obtaining the order of approximation of $f^{(k)}(x)$ by certain linear combinations of $L_{n,x}^{(k)} \bar{f}$.

Theorem 6 — If f is such that $\bar{f}$ and $f^{(2m+k)}(x)$ exist for some $x \in \mathbf{R}^+$, then

$$L_{n,x}^{(k)} \bar{f} - f^{(k)}(x) = \frac{d^k}{dx^k} \left\{ \sum_{r=1}^{2m} \frac{f^{(r)}(x)}{r!} \mu_{n,r}(x) \right\} + o(n^{-m}),\ n \to \infty \quad \ldots(3.8)$$

where $\mu_{n,r}(x) = L_{n,x}\overline{(t-x)^r}$. Further, if $f^{(2m+k)}(x)$ exists in $\langle a, b \rangle \subset \mathbf{R}^+$ and is continuous at each $x \in [a, b]$, then (3.8) holds uniformly in $x \in [a, b]$.

PROOF: If $f^{(2m+k)}(x)$ exists, then

$$f(t) = \sum_{r=0}^{2m+k} \frac{f^{(r)}(x)}{r!} (t-x)^r + h(t; x) \quad \ldots(3.9)$$

where $h(t; x)$ is such that $\bar{h}(t; x)$ exists and given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|h(t; x)| \leqslant \epsilon\, |t - x|^{2m+k},\ |t - x| < \delta. \quad \ldots(3.10)$$

If $f^{(2m+k)}(x)$ exists in $\langle a, b \rangle$ and is continuous at each $x \in [a, b]$, δ can be chosen independently of each $x \in [a, b]$. First, we shall show that

$$L_{n,x}^{(k)} \bar{h} = o(n^{-m}),\ n \to \infty \quad \ldots(3.11)$$

and that (3.11) holds uniformly for $x \in [a, b]$ in the uniformity case. We have

$$L_{n,x}^{(k)} \bar{h} = \left\{ \frac{\partial^k}{\partial y^k} \left[\frac{1}{(n+p)!} \left(\frac{n}{y} \right)^{n+p+1} \int_{I_\delta(x)} h(t; x)\, t^{n+p} e^{-nt/y}\, dt \right] \right\}_{y=x}$$

$$+ \left\{ \frac{\partial^k}{\partial y^k} \left[\frac{1}{(n+p)!} \left(\frac{n}{y} \right)^{n+p+1} \int_{J_\delta(x)} h(t; x)\, t^{n+p} e^{-nt/y}\, dt \right] \right\}_{y=x}. \quad \ldots(3.12)$$

The first term on the right-hand side of (3.12) is of $o(n^{-m})$ $(n \to \infty)$ by Lemma 1, (3.10) and eqn. (2.10). An application of Lemmas 1 and 2 shows that the second term on the right-hand side of (3.12) is also of $o(n^{-m})$ $(n \to \infty)$. This establishes (3.11).

In view of (3.11), to prove the theorem it is sufficient to show that

$$L_{n,x}^{(k)} \bar{g} = \frac{d^k}{dx^k} \left\{ \sum_{r=0}^{2m} \frac{f^{(r)}(x)}{r!} \mu_{n,r}(x) \right\} + o(n^{-m}) \quad \ldots(3.13)$$

and that (3.13) holds uniformly for each $x \in [a, b]$ in the uniformity case, where

$$g(t) = \sum_{r=0}^{2m+k} \frac{f^{(r)}(x)}{r!}(t-x)^r.$$

For this, if $Q(t)$ is a polynomial in t, by Theorem 1,

$$L_{n,x}\bar{Q} = \sum_{r=0}^{2m} \frac{Q^{(r)}(x)}{r!}\mu_{n,r}(x) + o(n^{-m}),\ n \to \infty \qquad \dots(3.14)$$

where the relation is also uniform in $x \in [a, b]$. By (2.7) it follows that $L_{n,x}\bar{Q}$ is also a polynomial of the same degree as $Q(t)$ and that the small order term on the right-hand side of (3.14) consists of a finite sum of terms like

$$\frac{P_{m+1}(x)}{n^{m+1}}, \frac{P_{m+2}(x)}{n^{m+2}}, \dots$$

where $P_{m+1}(x)$, $P_{m+2}(x)$ ... are polynomials in x. Therefore

$$L^{(k)}_{n,x}\bar{Q} = \frac{d^k}{dx^k}\left\{\sum_{r=0}^{2m} \frac{Q^{(r)}(x)}{r!}\mu_{n,r}(x)\right\} + o(n^{-m}),\ n \to \infty \qquad \dots(3.15)$$

which also holds uniformly in $x \in [a, b]$. Taking

$$Q(t) = \sum_{i=0}^{2m+k} \frac{f^{(i)}(x)}{i!}(t-x)^i \quad \text{in (3.15)},$$

and keeping in mind the boundedness of the derivatives of f we, thus, get (3.13) and also that it holds uniformly in the uniformity case. This completes the proof of the theorem.

4. Linear Combinations and Simultaneous Approximation

By Theorem 5 it is clear that a smoothness of f beyond the existence of $f^{(k+2)}(x)$ does not result in an improved approximation of $f^{(k)}(x)$ by $L^{(k)}_{n,x}\bar{f}$. Regarding the linear combinations $L^{(k)[m]}_{n,x}$, however, we have

Theorem 7 — Let f be such that $\bar{f}$ and $f^{(2m+k)}(x)$ exist for some $x \in \mathbf{R}^+$, then

$$L^{(k)[m+1]}_{n,x}\bar{f} - f^{(k)}(x) = o(n^{-m}) \text{ as } n \to \infty \qquad \dots(4.1)$$

and

$$L_{n,x}^{(k)[m]} \bar{f} - f^{(k)}(x) = O(n^{-m}) \text{ as } n \to \infty. \qquad ...(4.2)$$

Further, if $f^{(2m+k)}$ exists in $\langle a, b \rangle \subset \mathbb{R}^+$ and is continuous at each $x \in [a, b]$, then (4.1) and (4.2) hold uniformly in $x \in [a, b]$.

The proof of Theorem 7 can be obtained by using the definition of $L_{n,x}^{(k)[m]} \bar{f}$ and Theorem 6.

Theorem 8 — Let $k \in \mathbb{N}$, m be a non-negative integer and $0 \leqslant p' \leqslant 2m + 2$. If f is such that $\bar{f}(t)$ exists for some $t \in \mathbb{R}^+$ and $f^{(p'+k)}$ exists and is continuous on $\langle a, b \rangle \subset \mathbb{R}^+$, then for all $x \in [a, b]$ and n sufficiently large there holds

$$| L_{n,x}^{(k)[m+1]} \bar{f} - f^{(k)}(x) | \leqslant \max \left\{ \frac{c_k}{n^{p'/2}} \omega(f^{(p'+k)}; n^{-1/2}), \frac{c_k'}{n^{m+1}} \right\} \qquad ...(4.3)$$

where $c_k = c_k(m)$ and $c_k'(m; f)$ are constants and $\omega(f^{(p'+k)}; \delta)$ denotes the modulus of continuity of $f^{(p'+k)}$ on $\langle a, b \rangle$.

PROOF : With the hypothesis on f, for all $x \in [a, b]$ and $t \in \mathbb{R}^+$ we can write

$$f(t) = \sum_{i=0}^{p+k} \frac{f^{(i)}(x)}{i!} (t - x)^i + \frac{(t-x)^{p'+k}}{(p' + k)!} \{f^{(p'+k)}(\eta)$$

$$- f^{(p'+k)}(x)\} \chi_{<a,b>} (t) + (1 - \chi_{<a,b>} (t)) h(t; x) \qquad ...(4.4)$$

where $\chi_{<a,b>}$ denotes the characteristic function of $\langle a, b \rangle$, η is some number lying between t and x and $h(t; x)$ is a certain function such that $\bar{h}(t; x)$ exists. By the definition of $L_{n,x}^{(k)[m]} \bar{f}$, we have for some constants B_j

$$L_{n,x}^{(k)[m+1]} \bar{f} - f^{(k)}(x) = \sum_{j=0}^{m} B_j [L_{\alpha_j n,x}^{(k)} \bar{f} - f^{(k)}(x)].$$

Thus, defining

$$T_{x,\alpha_j n}(t) = \frac{1}{(\alpha_j n + p)!} \left(\frac{\alpha_j n}{x}\right)^{\alpha_j n+p+1} t^{\alpha_j n+p} e^{-\alpha_j nt/x} . Q_{k,j}(x) x^{-\alpha_j n-p-2k-1}$$

where

$$Q_{k,j}(x) = \sum_{\substack{i_1, i_2 \geqslant 0 \\ 2i_1+i_2 \leqslant k}} (\alpha_j n)^{i_1+i_2} (t - x)^{i_2} q_{i_1 i_2 k}(x)$$

by (4.4) and Lemma 1, if $g(t) = \sum_{i=1}^{p'+k} \frac{f^{(i)}(x)}{i!}(t-x)^i$, we have

$$L_{n,x}^{(k)[m+1]} \bar{f} - f^{(k)}(x) = L_{n,x}^{(k)[m+1]} \bar{g} + \sum_{j=0}^{m} B_j \int_0^\infty \frac{(t-x)^{p'+k}}{(p'+k)!}$$

$$\times [f^{(p'+k)}(\eta) - f^{(p'+k)}(x)] T_{x,\alpha_j n}(t)$$

$$\times \chi_{<a,b>}(t)\, dt + \sum_{j=0}^{m} B_j \int_0^\infty h(t; x)$$

$$\times (1 - \chi_{<a,b>}(t)) T_{x,\alpha_j n}(t)\, dt$$

$$= L_{n,x}^{(k)[m+1]} \bar{g} + \Sigma_1 + \Sigma_2, \text{ say.} \qquad \ldots(4.5)$$

From Definition 1, eqn. (2.7), Lemma 3 and Theorem 6 one has

$$| L_{n,x}^{(k)[m+1]} \bar{g} | \leqslant c_1 n^{-(m+1)} \qquad \ldots(4.6)$$

where c_1 is a constant depending on the maximum modulii of various derivatives $f^{(i)}(x)$ on $[a, b]$ and is independent of n.

To evaluate Σ_1 we proceed as follows:

$$\int_0^\infty \frac{| t - x |^{p'+k}}{(p'+k)!} | f^{(p'+k)}(\eta) - f^{(p'+k)}(x) | \chi_{<a,b>}(t) | T_{x,\alpha_j n}(t) | dt$$

$$\leqslant \frac{\omega(f^{(p'+k)}; \delta)}{(p'+k)!} \left\{ \int_0^\infty | t - x |^{p'+k} | T_{x,\alpha_j n}(t) | dt \right.$$

$$\left. + \frac{1}{\delta} \int_0^\infty | t - x |^{p'+k+1} | T_{x,\alpha_j n}(t) | dt \right\}.$$

Using (2.10), an analysis similar to that of the proof of Theorem 6 shows that the last expression does not exceed

$$\frac{\omega(f^{(p'+k)}; \delta)}{(p'+k)!} \left\{ \frac{A_{k+p'}}{(\alpha_j n)^{p'/2}} + \frac{A'_{k+p'}}{\delta(\alpha_j n)^{(p'+1)/2}} \right\}$$

where $A_{k+p'}$ and $A'_{k+p'}$ are constants independent of $\alpha_j n$ and f. Therefore, with $\delta = n^{-1/2}$, we have

$$\left|\sum\nolimits_1\right| \leqslant \frac{c_2}{n^{p'/2}}\,\omega(f^{(p'+k)};\,n^{-1/2}) \qquad \ldots(4.7)$$

for some constant c_2 not depending on n.

Finally, in view of Lemma 2, it is clear that $\Sigma_2 = o(n^{-s})$ uniformly in $x \in [a, b]$ for an arbitrary $s > 0$. Hence there exists a constant c_3 depending on f but independent of n such that

$$\left|\sum\nolimits_2\right| \leqslant \frac{c_3}{n^{m+1}}. \qquad \ldots(4.8)$$

The estimates (4.6) – (4.8) prove the theorem.

5. Inverse and Saturation Theorems

Theorem 9 (*Inverse theorem*) — Let f be such that $\bar{f}(t)$ exists for some $t \in \mathbf{R}^+$. Let $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$ and $0 < \alpha < 2$. Then in the following the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ 4 hold, provided (i) is meaningful i.e., $f^{(k)}(x)$ exists on $[a_1, b_1]$.

(1) $$\sup_{x\in[a_1,b_1]} |\, L^{(k)[m+1]}_{n_p,x}\,\bar{f} - f^{(k)}(x)\,| = O(n_p^{-\alpha(m+1)/2}),$$

$$\left(\frac{n_{p+1}}{n_p} \leqslant c, \text{ for some } c > 0\right);$$

(2) $f^{(k)}(x) \in \mathrm{Liz}\,(\alpha, m+1; a_2, b_2)$;

(3) (i) for $m' < \alpha\,(m+1) < m'+1$, $m' = 0, 1, 2, \ldots, 2m+1$, $f^{(k+m')}(x)$ exists and belongs to $\mathrm{Lip}\,(\alpha(m+1) - m'; a_2, b_2)$,

(ii) for $\alpha(m+1) = m'+1$, $m' = 0, 1, 2, \ldots, 2m$, $f^{(k+m')}(x)$ exists and belongs to $Z_1(a_2, b_2)$;

(4) $\| L^{(k)[m+1]}_{n,x}\,\bar{f} - f^{(k)} \|_{C[a_3,b_3]} = O(n^{-\alpha(m+1)/2})$.

Proof : Assume (1). Since $L^{(k)[m+1]}_{n,x}\,\bar{f}$ are continuous functions and by (1) they converge to $f^{(k)}(x)$ uniformly on $[a_1, b_1]$, it follows that $f^{(k)}(x)$ is continuous in the relative topology of $[a_1, b_1]$. Let a_1', b_1', a_1'' and b_1'' satisfy $a_1 < a_1'' < a_1' < a_2$, $b_2 < b_1' < b_1'' < b_1$. Let $f_* \in C_0^k(\mathbf{R}^+)$ with Supp. $f_* \subset (a_1, b_1)$ such that $f_*(x) = f(x)$ for $x \in (a_1'', b_1'')$. The existence of such a function is trivial. Now, in view of Lemmas 1 and 2 it is clear that

$$\| L^{(k)[m+1]}_{n,x}\,[\bar{f} - \bar{f}_*] \|_{C[a_1',b_1']} = o(n^{-s})$$

for an arbitrary $s > 0$. It follows that

$$\| L_{n,x}^{(k)[m+1]} \bar{f}_* - f_* \|_{C[a_1', b_1']} = O(n^{-\alpha(m+1)/2}).$$

In view of the validity of (2.14) for f_*, Theorem 9 follows from the continuous version of Theorem 2 and the intermediate steps in its proof, since $L_{n,x}^{(k)} \bar{f}_* \equiv x^{-k} L_{n,x}\bar{I}$, where $g(t) \equiv t^k f_*^{(k)}(t)$.

By arguments similar to those considered for the proof of Theorem 9, from the continuous version of Theorem 3 and its proof there follows the saturation theorem:

Theorem 10 (*saturation theorem*) — Let f be such that $\bar{f}(t)$ exists for some $t \in \mathrm{R}^+$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then in the following (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) hold, provided (1) is meaningful, i.e., $f^{(k)}(x)$ exists on $[a_1, b_1]$.

(1) $n_p^{m+1} \sup_{x\in[a_1,b_1]} | L_{n_p,x}^{(k)[m+1]} \bar{f} - f^{(k)}(x) | = O(1) \left(\frac{n_{p+1}}{n^p} \leqslant c \text{ (for some } c > 0\right)$;

(2) $f^{(k+2m+1)} \in A.C.[a_2, b_2]$ and $f^{(k+2m+2)} \in L_\infty[a_2, b_2]$;

(3) $n^{m+1} \| L_{n,x}^{(k)[m+1]} \bar{f} - f^{(k)} \|_{C[a_3,b_3]} = O(1)$.

(4) $n_p^{m+1} \sup_{x\in[a_1,b_1]} | L_{n_p,x}^{(k)[m+1]} \bar{f} - f^{(k)}(x) | = o(1)$;

(5) $f^{(k)}(x) \in C^{2m+2}[a_2, b_2]$ and $\sum_{i=\max(m,k)}^{2m+2+k} Q(i, m, x) f^{(i)}(x) = 0, x \in [a_2, b_2]$,

where $Q(i, m, x)$ are certain polynomials depending on m and k;

(6) $n^{m+1} \| L_{n,x}^{(k)[m+1]} \bar{f} - f^{(k)} \|_{C[a_3,b_3]} = o(1)$.

Finally, we observe that to complete the proofs of Theorems 2 and 3 for a general function f, it is sufficient to work with an auxiliary function $f_* \in C_0^k(\mathrm{R}^+)$ with supp. $f_* \subset (a_1, b_1)$ and coinciding with f on (a_1'', b_1'') and to proceed in the manner of the proof of Theorem 9.

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