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PROPERTY (I) DOES NOT IMPLY SMOOTHNESS

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A Banach space X is said to have property (I) if every closed convex subset of X can be represented as the intersection of closed balls. Pethe and Thakare (1977) posed several problems one of which was: If X is a Banach space that has property (I) need X be smooth?

We answer in the negative this problem by modifying a result of Brown (1974). We construct an example of a three dimensional Banach space whose norm satisfies the property (I) but which is not smooth.

§1. Let X be a Banach space. We say that X has *property* (I) if every bounded closed convex subset of X can be represented as the intersection of closed balls that contain it. It is known that if a Banach space is Frechet differentiable then it satisfies property (I) (see Pethe and Thakare 1977, Sullivan 1977). A Banach space is said to be very smooth if there exists a support mapping from X to X^* that is norm to weak continuous. Frechet differentiability implies very smoothness of the space. In the context of this it is worthwhile to ask:

If X is a Banach space that has property (I) need X be smooth?

This problem seems to be legitimate on the background of a result (Phelps 1960) that runs as: A two dimensional normed linear space has property (I) if and only if X is smooth.

The purpose of this note is to put to rest the possibility of a Banach space (even finite dimensional) with property (I) to be smooth.

§2. We begin with a lemma that is crucial for our discussion. This result is a slight restatement of a lemma proved by Brown (1974); and its proof is essentially due to him.

Lemma : Let X be the real linear space and let p_1 and p_2 be two equivalent norms on X with respect to which X is separable. If $p_1(x) \leqslant p_2(x)$ for all x in X then there exists an equivalent norm p on X with $p_1(x) \leqslant p(x) \leqslant p_2(x)$ for all x in

X and if $p(x) = 1$, $p_1(x) < p_2(x)$, then $x \in \text{ext } U_p$, where ext U_p is the set of extreme points of the unit-p-ball of X.

PROOF : The proof of the lemma runs exactly on the same lines as given in Brown (1974) up to the definition of norm p.

Define p by

$$p(x) = \sum_{n=1}^{\infty} 2^{-n} p_{x(n)}(x).$$

This p is clearly a norm on X that satisfies the conditions of the Lemma. Suppose on the contrary, $p(x) = 1$, $p_1(x) < p_2(x)$ and $x \notin \text{ext } U_p$, the set of all extreme points of the unit-p-ball. Now if x is an extreme point of U_p then there must be a line segment on the unit sphere such that this line segment contains the point x. Since $P_1(x) < P_2(x)$, $x \in V$ and V is open, there must be a part of this line segment containing x that is completely contained in V. It now follows that, for some m, the set

$$V_{x(m)} \cap \{x : p(x) = 1\}$$

contains a nondegenerate line segment $[y, z]$ say.

For the remaining part of the proof refer to Brown (1974).

§3. *A counter exemple* — Consider $X = R^3$. Let p_2 be the sup-norm on X. The p_1-norm on X is defined in terms of its unit ball. Let

$$\begin{aligned} &\text{Unit-}p_1\text{-ball} \\ &= \text{Conv}\,(U_{p_2} \cup \{\pm y_1, \pm y_2, \pm y_3\}) \end{aligned}$$

where U_{p_2} is the unit-p_2-ball,

$$y_1 = (1 + \delta, 0, 0), y_2 = (0, 1 + \delta, 0), y_3 = (0, 0, 1 + \delta)$$

with δ sufficiently small. Note that $p_1(x) \leqslant p_2(x)$. It is a classical result that all norms on finite-dimensional Banach spaces are equivalent. By Lemma we conclude that there exists an equivalent norm 'p' on X with the property that $p_1(x) \leqslant p(x) \leqslant p_2(x)$ for all x in X. Further, if $p(x) = 1$, $p_1(x) < p_2(x)$, then $x \in \text{ext } U_p$, the set of extreme points of the unit-p-ball. Hence we conclude that with respect to norm 'p' the extreme points are dense on the unit sphere but they are not equal. By Theorem 4.4 of Phelps (1960) we conclude that the predual of (X, p) with appropriate norm (note that such a norm exists because of reflexivity of the space) satisfies property (I), but it is not smooth (p being not rotund). This in fact finishes our programme.

§4. *Open problem* — We do not know whether the converse of the above phenomenon holds? That is, whether very smoothness of the Banach space implies the property (I)? In particular whether reflexive smooth spaces have property (I)?

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