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# FIXED POINT THEOREMS IN TYCHONOFF SPACES

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Extensions of Banach's contraction principle are considered in the non-metric setting of a topology generated by a family of pseudometrics. A class of mappings on such spaces, called contingent contractions, is defined and used to prove a fixed point theorem from which several recent results follow as corollaries.

For selfmappings $T$ of a complete metric space $(X, d)$ that satisfy the condition $d(Tx, Ty) < qd(x, y)$ for all $x, y \in X$, $x \neq y$, $q$ being a constant $0 \leqslant q < 1$, a well-known theorem of Banach (1922) states that there exists a unique point $\xi \in X$ such that $T\xi = \xi$. The usefulness of this "principle of contraction mappings" in analysis has been well illustrated by Kolmogorov and Fomin (1957, pp. 43–51).

In this paper we consider a double generalization of this theorem. Firstly the contractive nature of the mapping is generalized after the style of Ćirić (1972), and secondly the underlying space is freed to a non-metrizable situation. The topological space $(X, \mathcal{T})$ is called a gauge space if the topology $\mathcal{T}$ is generated by a family $\{d_\alpha : \alpha \in \Gamma\}$ of pseudometrics on $X$, in the sense that the family of balls

$$\{B(x, d_\alpha, \epsilon) : x \in X, \alpha \in \Gamma, \epsilon > 0\}$$

where $B(x, d_\alpha, \epsilon) = \{y \in X : d_\alpha(x, y) < \epsilon\}$ is a subbase for $\mathcal{T}$. It is well known that $(X, \mathcal{T})$ is a gauge space if and only if it is completely regular. Furthermore, $(X, \mathcal{T})$ is Hausdorff (and hence Tychonoff) if and only if for $x, y \in X$, $x \neq y$ there is an $\alpha \in \Gamma$ such that $d_\alpha(x, y) > 0$ [see Dugundji (1966, pp. 198–200)]. Tan (1972) has considered fixed point theorems in this non-metric setting, so that this paper may be regarded as a continuation of his work. Throughout this paper, unless otherwise stated, $X$ is a gauge space generated by the family $\{d_\alpha : \alpha \in \Gamma\}$ of pseudometrics.

If $\{x_n : n = 0, 1, 2, \ldots\}$ is a sequence in $X$ and $x \in X$, then $\{x_n\}$ converges to $x$, written as $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$, if and only if for each $\alpha \in \Gamma$, $d_\alpha(x, x_n) \to 0$ as $n \to \infty$. The sequence $\{x_n : n = 0, 1, 2, \ldots\}$ in $X$ is said to be Cauchy if and only if for each $\alpha \in \Gamma$, $d_\alpha(x_n, x_m) \to 0$ as $n, m \to \infty$.

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*Definition* 1 — A selfmapping $T$ of a gauge space $(X, \mathcal{T})$ is a contingent contraction (henceforth abbreviated as c. contraction) if for each $\alpha \in \Gamma$ there exist non-negative bounded real valued functions $q_\alpha$, $r_\alpha$, $s_\alpha$, $t_\alpha$ and $u_\alpha$ defined on $X \times X$ such that for all $x, y \in X$ we have

$$d_\alpha(Tx, Ty) \leqslant q_\alpha(x, y)\, d_\alpha(x, y) + r_\alpha(x, y)\, d_\alpha(x, Tx) + s_\alpha(x, y)\, d_\alpha(y, Ty) + t_\alpha(x, y)\, d_\alpha(x, Ty) + u_\alpha(x, y)\, d_\alpha(y, Tx)$$

and either

(i) $$\limsup_{n\to\infty} \sup \left\{ \left[ \frac{q_\alpha(T^n x, T^{n+1}x) + r_\alpha(T^n x, T^{n+1}x) + t_\alpha(T^n x, T^{n+1}x)}{1 - s_\alpha(T^n x, T^{n+1}x) - t_\alpha(T^n x, T^{n+1}x)} \right] : x \in X \right\} = \lambda_\alpha < 1$$

and $$\limsup_{n\to\infty} \sup\{s_\alpha(T^n x, y) + t_\alpha(T^n x, y) : x, y \in X\} < 1$$

or

(i)′ $$\limsup_{n\to\infty} \sup \left\{ \left[ \frac{q_\alpha(T^{n+1}x, T^n x) + s_\alpha(T^{n+1}x, T^n x) + u_\alpha(T^{n+1}x, T^n x)}{1 - r_\alpha(T^{n+1}x, T^n x) - u_\alpha(T^{n+1}x, T^n x)} \right] : x \in X \right\} = \lambda'_\alpha < 1$$

and $$\limsup_{n\to\infty} \sup\{r_\alpha(y, T^n x) + u_\alpha(y, T^n x) : x, y \in X\} < 1.$$

This notion is a generalization of the concept of $\lambda$-generalized contraction for metric spaces introduced by Ćirić (1972, Definition 2.1). If $X$ is a gauge space, a selfmapping $T$ on $X$ is called a $\lambda$-generalized contraction if for each pair of points $x, y \in X$ and for each $\alpha \in \Gamma$ there are non-negative numbers $q_\alpha(x, y)$, $r_\alpha(x, y)$, $s_\alpha(x, y)$, $t_\alpha(x, y)$ such that

$$\sup\{q_\alpha(x, y) + r_\alpha(x, y) + s_\alpha(x, y) + 2t_\alpha(x, y) : x, y \in X, \alpha \in \Gamma\} = \lambda < 1$$

and $$d_\alpha(Tx, Ty) \leqslant q_\alpha(x, y)\, d_\alpha(x, y) + r_\alpha(x, y)\, d_\alpha(x, Tx) + s_\alpha(x, y)\, d_\alpha(y, Ty) + t_\alpha(x, y)\, \{d_\alpha(x, Ty) + d_\alpha(y, Tx)\}.$$

The following example shows that for metric spaces there are c. contractions which are not $\lambda$-generalized contractions.

*Example* — Let $X = [0, 10] \cup \{11.5, 12\}$ regarded as a subspace of the real line with the usual metric. Define the mapping $T$ by $Tx = \frac{3}{4}x$ for all $x \in X - \{12\}$, and $T(12) = 11.5$. Then $T$ is a c. contraction. For if $x, y \neq 12$ take

$$q(x, y) = \frac{3}{4},\ r(x, y) = s(x, y) = t(x, y) = u(x, y) = \frac{1}{20},$$

$$q(x, 12) = \frac{3}{4},\ r(x, 12) = s(x, 12) = t(x, 12) = \frac{1}{20},\ u(x, 12) = \frac{2}{3},$$

and $$q(12, x) = \frac{3}{4}, r(12, x) = s(12, x) = u(12, x) = \frac{1}{20}, t(12, x) = \frac{2}{3}.$$

However, $T$ is not a $\lambda$-generalized contraction, for consider the points $x = 10$, $y = 12$.

*Definition* 2 — Let $T$ be a selfmapping of a gauge space $(X, \mathcal{T})$. Then $X$ is said to be $T$-orbitally complete if every sequence of the form $\{T^{n(i)}(x) : i = 0, 1, 2, ...\}$ where $x \in X$, which is a Cauchy sequence has a limit point in $X$.

A gauge space $X$ is sequentially complete [see Tan (1972, Definition 1.6)] if every Cauchy sequence in $X$ converges to some point of $X$. If $X$ is a gauge space, then $X$ is sequentially compact implies that $X$ is countably compact which implies that $X$ is sequentially complete which implies that $X$ is $T$-orbitally complete for any selfmapping $T$ of $X$. It is possible for a non-complete gauge space to be orbitally complete with respect to one selfmapping but not another. [See Ciric (1971, Example 3) for an example in the metric case].

*Theorem* 1 — If $T$ is a c. contraction of a $T$-orbitally complete Hausdorff gauge space $X$, then $T$ has a fixed point in $X$. Moreover, this fixed point is unique if

$$\sup \{q_\alpha(x, y) + t_\alpha(x, y) + u_\alpha(x, y) : x, y \in X, \alpha \in \Gamma\} < 1.$$

PROOF: Let $x_0$ be any point of $X$ and define the sequence

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots x_n = Tx_{n-1} = T^n x_0, \dots .$$

Let $\alpha \in \Gamma$, and suppose that condition (i) of Definition 1 is satisfied. Then we have

$$\begin{aligned} d_\alpha(x_n, x_{n+1}) &= d_\alpha(Tx_{n-1}, Tx_n) \leqslant q_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, x_n) \\ &\quad + r_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, Tx_{n-1}) + s_\alpha(x_{n-1}, x_n)\, d_\alpha(x_n, Tx_n) \\ &\quad + t_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, Tx_n) + u_\alpha(x_{n-1}, x_n)\, d_\alpha(x_n, Tx_{n-1}) \\ &= q_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, x_n) + r_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, x_n) \\ &\quad + s_\alpha(x_{n-1}, x_n)\, d_\alpha(x_n, x_{n+1}) + t_\alpha(x_{n-1}, x_n)\, d_\alpha(x_{n-1}, x_{n+1}). \end{aligned}$$

Using the triangle inequality we have

$$\begin{aligned} d_\alpha(x_n, x_{n+1}) &\leqslant \{q_\alpha(x_{n-1}, x_n) + r_\alpha(x_{n-1}, x_n)\}\, d_\alpha(x_{n-1}, x_n) \\ &\quad + s_\alpha(x_{n-1}, x_n)\, d_\alpha(x_n, x_{n+1}) + t_\alpha(x_{n-1}, x_n)\, \{d_\alpha(x_{n-1}, x_n) \\ &\quad + d_\alpha(x_n, x_{n+1})\}. \end{aligned}$$

Therefore

$$d_\alpha(x_n, x_{n+1}) \leqslant \frac{q_\alpha(x_{n-1}, x_n) + r_\alpha(x_{n-1}, x_n) + t_\alpha(x_{n-1}, x_n)}{1 - s_\alpha(x_{n-1}, x_n) - t_\alpha(x_{n-1}, x_n)}\, d_\alpha(x_{n-1}, x_n).$$

Thus, there is a positive integer $N$ such that

$$d_\alpha(x_{N+n}, x_{N+n+1}) \leqslant \lambda_\alpha d_\alpha(x_{N+n-1}, x_{N+n}) \leqslant \lambda_\alpha^n \, d_\alpha(x_N, x_{N+1}).$$

Hence $d_\alpha(x_n, x_m) \to 0$ as $n, m \to \infty$, since $\lambda_\alpha < 1$.

If, on the other hand, condition (i)′ of Definition 1 is satisfied, then by considering $d_\alpha(x_{n+1}, x_n)$ we have that $d_\alpha(x_n, x_m) \to 0$ as $n, m \to \infty$. Thus in both cases $\{x_n\}$ is a Cauchy sequence, and since $X$ is $T$-orbitally complete, there is a point $\xi$ in $X$ such that $\xi = \lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} x_n$.

We claim that $\xi$ is a fixed point of $T$ in $X$. In case (i) we have

$$\begin{aligned} d_\alpha(Tx_n, T\xi) \leqslant\ & q_\alpha(x_n, \xi)\, d_\alpha(x_n, \xi) + r_\alpha(x_n, \xi)\, d_\alpha(x_n, x_{n+1}) \\ & + s_\alpha(x_n, \xi)\, \{d_\alpha(\xi, x_{n+1}) + d_\alpha(x_{n+1}, T\xi)\} \\ & + t_\alpha(x_n, \xi)\, \{d_\alpha(x_n, x_{n+1}) + d_\alpha(x_{n+1}, T\xi)\} \\ & + u_\alpha(x_n, \xi)\, d_\alpha(\xi, x_{n+1}). \end{aligned}$$

Hence

$$\begin{aligned} \{1 - s_\alpha(x_n, \xi) - t_\alpha(x_n, \xi)\}\, d_\alpha(Tx_n, T\xi) \leqslant\ & q_\alpha(x_n, \xi)\, d_\alpha(x_n, \xi) \\ & + \{r_\alpha(x_n, \xi) + t_\alpha(x_n, \xi)\}\, d_\alpha(x_n, x_{n+1}) + \{s_\alpha(x_n, \xi) \\ & + u_\alpha(x_n, \xi)\}\, d_\alpha(\xi, x_{n+1}). \end{aligned}$$

Now there is a positive integer $N'$ such that

$$\sup\{s_\alpha(x_n, y) + t_\alpha(x_n, y) : x, y \in X\} < 1 \quad \text{for} \quad n > N'.$$

Moreover, the functions $q_\alpha$, $r_\alpha$, $s_\alpha$, $t_\alpha$ and $u_\alpha$ are bounded, so that $d_\alpha(Tx_n, T\xi) \to 0$ as $n \to \infty$.

In case (i)′, by considering $d_\alpha(T\xi, Tx_n)$ we show that $d_\alpha(T\xi, Tx) \to 0$ as $n \to \infty$. Thus for all $\alpha \in \Gamma$, $d_\alpha(T\xi, Tx_n) \to 0$ as $n \to \infty$, so that $T\xi = \lim_{n\to\infty} x_n$. But $X$ is Hausdorff and $\xi = \lim_{n\to\infty} x_n$, so that $T\xi = \xi$ as desired.

If $\eta \in X$ and $\eta = T\eta$ then we have for all $\alpha \in \Gamma$ that

$$\begin{aligned} d_\alpha(\xi, \eta) = d_\alpha(T\xi, T\eta) \leqslant\ & q_\alpha(\xi, \eta)\, d_\alpha(\xi, \eta) + t_\alpha(\xi, \eta)\, d_\alpha(\xi, \eta) \\ & + u_\alpha(\xi, \eta)\, d_\alpha(\eta, \xi), \end{aligned}$$

that is

$$d_\alpha(\xi, \eta) \leqslant \{q_\alpha(\xi, \eta) + t_\alpha(\xi, \eta) + u_\alpha(\xi, \eta)\}\, d_\alpha(\xi, \eta).$$

If $\xi \neq \eta$ there is a $\beta \in \Gamma$ such that $d_\beta(\xi, \eta) > 0$.

By hypothesis $q_\beta(\xi, \eta) + t_\beta(\xi, \eta) + u_\beta(\xi, \eta) < 1$, so that we have the contradiction

$$d_\beta(\xi, \eta) < d_\beta(\xi, \eta).$$

We now state the preceding result for a metric space. This seems to be a new result.

*Corollary* — If $T$ is a c. contraction of a $T$-orbitally complete metric space $X$, then $T$ has a fixed point in $X$ which will be unique if

$$\sup \{q(x, y) + t(x, y) + u(x, y) : x, y \in X\} < 1.$$

The metric version of the next theorem has been proven by Ćirić (1972, Theorem 2.6). It follows immediately from Theorem 1 by observing that a $\lambda$-generalized contraction is a c. contraction such that

$$\sup \{q_\alpha(x, y) + t_\alpha(x, y) + u_\alpha(x, y) : x, y \in X, \alpha \in \Gamma\} < 1$$

since $t_\alpha(x, y) = u_\alpha(x, y)$ for all $x, y \in X$ and $\alpha \in \Gamma$.

*Theorem* 2 — If $T$ is a $\lambda$-generalized contraction of a $T$-orbitally complete Hausdorff gauge space, then $T$ has a unique fixed point in $X$.

The following result is an extension to gauge spaces of a theorem proved by Zamfirescu (1972, Theorem 1) for metric spaces.

*Theorem* 3 — Let $X$ be a sequentially complete Hausdorff gauge space such that for each $\alpha \in \Gamma$ there are non-negative real numbers $a_\alpha$, $b_\alpha$, $c_\alpha$ which satisfy

$$\sup \{a_\alpha : \alpha \in \Gamma\} < 1, \sup \{b_\alpha : \alpha \in \Gamma\} < \tfrac{1}{2} \text{ and } \sup \{c_\alpha : \alpha \in \Gamma\} < \tfrac{1}{2}.$$

Let $T$ be a selfmapping of $X$ such that for each pair of points $x, y \in X$ and each $\alpha \in \Gamma$ at least one of the following conditions is satisfied:

(1) $d_\alpha(Tx, Ty) \leqslant a_\alpha d_\alpha(x, y)$,

(2) $d_\alpha(Tx, Ty) \leqslant b_\alpha\{d_\alpha(x, Tx) + d_\alpha(y, Ty)\}$,

(3) $d_\alpha(Tx, Ty) \leqslant c_\alpha\{d_\alpha(x, Ty) + d_\alpha(y, Tx)$.

Then $T$ has a unique fixed point in $X$.

The proof follows by noting that such a mapping $T$ is a c. contraction of $X$ and appealing to Theorem 1.

The next three results follow immediately from Theorem 3. Theorem 4 is Banach's classical contraction principle for gauge spaces [see Tan (1972, Theorem 2.3)]. Theorems 5 and 6 have been proven in the metric case by Kannan (1968) and Zamfirescu (1972, Corollary 3) respectively.

*Theorem* 4 — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $a_\alpha$ with $0 \leqslant a_\alpha < 1$ and $d_\alpha(Tx, Ty) \leqslant a_\alpha d_\alpha(x, y)$ for all $x, y \in X$, then $T$ has a unique fixed point in $X$.

*Theorem* 5 — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $b_\alpha$ with $0 \leqslant b_\alpha < \frac{1}{2}$ and $d_\alpha(Tx, Ty) \leqslant b_\alpha\{d_\alpha(x, Tx) + d_\alpha(y, Ty)\}$ for all $x, y \in X$ then $T$ has a unique fixed point in $X$.

*Theorem* 6 — If $T$ is a selfmapping of a sequentially complete Hausdorff gauge space $X$ such that for each $\alpha \in \Gamma$ there is a real number $c_\alpha$ with $0 \leqslant c_\alpha < \frac{1}{2}$ and

$$d_\alpha(Tx, Ty) \leqslant c_\alpha\{d_\alpha(x, Ty) + d_\alpha(y, Tx)\}$$

then $T$ has a unique fixed point in $X$.

Theorems 4, 5 and 6 also follow immediately from the next result which generalizes a theorem of Hardy and Rogers (1973) for metric spaces.

*Theorem* 7 — Let $X$ be a sequentially complete Hausdorff gauge space and $T$ a selfmapping of $X$ such that for each $\alpha \in \Gamma$ there are non-negative real numbers $a_\alpha$, $b_\alpha$, $c_\alpha$, $e_\alpha$ and $f_\alpha$ such that

$$\sup\{a_\alpha + b_\alpha + c_\alpha + e_\alpha + f_\alpha : \alpha \in \Gamma\} = \lambda < 1$$

and

$$d_\alpha(Tx, Ty) \leqslant a_\alpha d_\alpha(x, y) + b_\alpha d_\alpha(x, Tx) + c_\alpha d_\alpha(y, Ty) + e_\alpha d_\alpha(x, Ty) + f_\alpha d_\alpha(y, Tx),$$

for all $x, y \in X$. Then $T$ has a unique fixed point in $X$.

Such a mapping $T$ is a c. contraction, since we may, without loss of generality, take $e_\alpha = f_\alpha$ for each $\alpha \in \Gamma$. Then

$$a_\alpha + b_\alpha + c_\alpha + e_\alpha + \lambda e_\alpha < \lambda \text{ yields } \frac{a_\alpha + b_\alpha + e_\alpha}{1 - c_\alpha - e_\alpha} < \lambda,$$

so that $T$ satisfies condition (i) of Definition 1.

By taking $e_\alpha = f_\alpha = 0$ for each $\alpha \in \Gamma$, we obtain a gauge space version of the theorem of Reich (1971a).

Another similar application of Theorem 1 gives the following result which extends to completely regular spaces the series of results of Rakotch (1962), Reich (1971b, Theorem 3) and Hardy and Rogers (1973, Theorem 2).

*Theorem* 8 — Let $X$ be a sequentially complete Hausdorff gauge space, and for each $\alpha \in \Gamma$ let $a_\alpha$, $b_\alpha$, $c_\alpha$, $e_\alpha$ and $f_\alpha$ be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$ such that $a_\alpha(t) + b_\alpha(t) + c_\alpha(t) + e_\alpha(t) + f_\alpha(t) < 1$. Suppose $T$ is a selfmapping of $X$ such that

$$\begin{aligned} d_\alpha(Tx, Ty) \leqslant a_\alpha(d_\alpha(x, y))\, d_\alpha(x, y) &+ b_\alpha(d_\alpha(x, y))\, d_\alpha(x, Tx) \\ &+ c_\alpha(d_\alpha(x, y))\, d_\alpha(y, Ty) + e_\alpha(d_\alpha(x, y))\, d_\alpha(x, Ty) \\ &+ f_\alpha(d_\alpha(x, y))\, d_\alpha(y, Tx) \end{aligned}$$

for all $x, y \in X$ and $\alpha \in \Gamma$. Then $T$ has a unique fixed point in $X$.

## REFERENCES

Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, **3**, 133–81.

Ćirić, L. B. (1971). On contraction type mappings. *Math. Balkanica*, **1**, 52–57.

——— (1972). Generalized contractions and fixed-point theorems. *Publ. Inst. Math. Nouv. série*, **13**, 11–16.

Dugundji, J. (1966). Topology. Allyn and Bacon, Boston.

Hardy, G. E., and Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. *Canad. Math. Bull.*, **16**, 201–206.

Kannan, R. (1968). Some results on fixed points. *Bull. Calcutta math. Soc.*, **60**, 71–76.

Kolmogorov, A. N., and Fomin, S. V. (1957). Elements of the Theory of Functions and Functional Analysis, Vol. 1. Graylock Press, Rochester, New York.

Rakotch, E. (1962). A note on contractive mappings. *Proc. Am. math. Soc.*, **13**, 459–65.

Reich, S. (1971a). Some remarks concerning contraction mappings. *Canad. Math. Bull.*, **14**, 121–24.

——— (1971b). Kannan's fixed point theorem. *Boll. Un. Mat. Ital.*, **4**, 1–11.

Tan, K. K. (1972). Fixed point theorems for non-expansive mappings. *Pacific J. Math.*, **41**, 829–42.

Zamfirescu, T. (1972). Fixed point theorems in metric spaces. *Arch. Math. (Basel)* **23**, 292–98.