Indian J. pure appl. Math., 11(5) : 572-575, May 1980

SOME BANACH ALGEBRA CONCEPTS, STATES OF OPERATORS AND APPROXIMATE SPECTRA

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(*Received* 12 *May* 1979)

In the present paper some results of Goldberg (1966) are reproved and a result of Nakamoto and Nakamura (1972) is generalized. Some theorems in Banach algebra are obtained and some probable results concerning them are indicated.

§1. The purpose of this brief note is to point out some generalizations and extensions of some results of Goldberg (1966) on the relationship between some Banach algebra concepts and the states of an operator. In section 3, we obtain a generalization of a result on approximate spectra of operators on a Hilbert space by Nakamoto and Nakamura (1972). For other results on approximate spectra see Kasahara and Takai (1972) and Enomoto *et al.* (1972). In section 4 we are able to obtain some theorems in Banach algebras although we do not have the full structure of an algebra and indicate some probable results concerning them.

§2. Throughout this section X and Y are normed linear spaces and T is a linear operator with domain $\mathcal{D}(T) \subseteq X$, range $\mathcal{R}(T) \subseteq Y$ and null space $\mathcal{N}(T)$.

We will use the same notation and conventions as in Goldberg (1966) and Taylor's classification for the state of an operator.

The following theorems are due to Goldberg (1966, pp. 72-75). However, we do not assume that T is closed or that $\mathcal{D}(T)$ is dense in X.

Theorem 2.1 — $III = D_r$.

PROOF : If $T \in III$, then $AT = 0$, where A is the Schatten (1950) cross product of a vector in $\mathcal{D}(T)$ and a functional in $\mathcal{R}(T)$.

Conversely, if $AT = 0$ on $\mathcal{D}(T)$ and if $\mathcal{R}(T)$ were dense in Y, we would have $A = 0$ on a dense set.

*Supported in part by a grant from the San Diego State University Foundation.

Theorem 2.2 — $1 = CZ_l$.

PROOF : Suppose $T \in 1$ and A is an inverse for T. Then if $TA_n \to 0$, we have $A_n = (AT) A_n = A(TA_n) \to 0$.

If $T \notin 1$, one can find $x_n \in X$ and $y' \in Y'$ all of norm 1 such that

$$\| TA_n \| \leqslant \| Tx_n \| \to 0$$

where $A_n = x_n \otimes y'$.

Theorem 2.3 — $3 = D_l$.

PROOF : The proof is similar to Goldberg (1966).

§3. We begin with some generalizations of some of Goldberg's theorems. If T is not 1-1, then one can define $\hat{T}$ on $\mathcal{D}(T)/\mathcal{N}(T)$ by $\hat{T}[x] = y$ where

$$[x] = \{x : x \in X, Tx = y\}.$$

It follows that $\hat{T}$ is 1-1 on $\mathcal{D}(T)/\mathcal{N}(T)$ and $\mathcal{R}(T) = \mathcal{R}(\hat{T})$. See Goldberg [1966, p. 63].

Theorem 3.1 — If $\mathcal{N}(T)$ is closed, then $D_l \cap R_r \subseteq \{T : T \in I_3$ and $\hat{T} \in I_1\} \subseteq D_l \cap CZ_r$.

PROOF : Suppose $T \in D_l \cap R_r$. Define $\hat{A}$ by $\hat{A}y = [Ay] \in \mathcal{D}(T)/\mathcal{N}(T)$ where $y \in Y$ and A is any right inverse for T. It follows easily that $\hat{A}$ is a two sided continuous inverse for $\hat{T}$.

To prove the second inclusion, suppose there exist $A_n \in [Y, X]$ such that $A_nT \to 0$. Then $\| A_n \| \leqslant \| A_n\hat{T} \| \, \| \hat{A} \| \to 0$.

Theorem 3.2 — If $\mathcal{D}(T)$ is dense in X, then $T \in CZ_r$ if and only if $T' \in 1$ if and only if $T' \in CZ_l$.

Corollary 3.3 (Goldberg 1966) — If X and Y are complete and T is closed with $\mathcal{D}(T)$ dense in X, then $I = CZ_r$.

Theorem 3.4 — $D_r \cap R_l = \{T : T \in III_1$ and T^{-1} has a continuous extension to all of $Y\}$.

PROOF : If $T \in III_1$ and T^{-1} has a continuous extension $A \in [Y, X]$, then for all $x \in \mathcal{D}(T)$, $(AT)\, x = A(Tx) = T^{-1}(Tx) = x$.

Conversely, suppose $T \in D_r \cap R_l$. Then there exists an $A \in [Y, X]$ such that $AT = I$ on $\mathcal{D}(T)$. A simple calculation shows that the restriction of A to the range of T is an inverse for T.

Corollary 3.5 — If X and Y are complete and T is closed with (T) dense in X, then $D_l \cap R_r = I_3$.

PROOF : Apply Theorem 3.4 to T' and then use the state diagram.

Corollary 3.6 — If X and Y are Hilbert spaces, then $D_r \cap CZ_l = D_r \cap R_l$.

Our next theorem generalizes a theorem concerning Banach algebras in Rickart (1960, p. 22).

Theorem 3.7 — $Z_r \cap R_l = D_r \cap R_l$.

PROOF : It suffices to show that $Z_r \cap R_l \subseteq D_r \cap R_l$. Toward this end, suppose there is an A such that $AT = I$ on $\mathcal{D}(T)$ and $\mathcal{R}(T)$ is dense in Y. Then for $y \in \mathcal{R}(T)$, $TAy = y$. Thus if $A_nT \to 0$, then on $\mathcal{R}(T)$ we have $A_n \to 0$.

Another Banach algebra type result is:

Theorem 3.8 — If X and Y are complete and T is closed with $\mathcal{D}(T)$ dense in X, then $R_r = CZ_r$.

PROOF : If $T \in CZ_r$, then $T \in I$. If $T \in I_1$, then $T \in R_l \cap R_r$. If $T \in I_3$, then $T \in D_l \cap R_r$. In either case, $T \in R_r$.

We now indicate how Theorem 1 in Nakamoto and Nakamura (1972) has been generalized. The equivalence of (i) and (ii) in Theorem 1 is a corollary of Theorem 2.2. The inequality $(T - \lambda)^* (T - \lambda) \geqslant \epsilon$ is equivalent

$$\| (T - \lambda) x \|^2 \geqslant \epsilon \| x \|^2,$$

for all x in X. Hence (i) and (iv) are equivalent. It T is coutinuous, then (iii) is seen to be equivalent to $T - \lambda \notin R_l$, so that we can prove (ii) equivalent to (iii) by showing $CZ_l = R_l$. This equality may fail to hold if $T \in III_1$ and T^{-1} fails to have a continuous extension to all of Y. However, if X and Y are Hilbert spaces, then $CZ_l = R_l$ is easily proved using $1 = CZ_l$ and Theorem 3.4.

§4. In this section, we consider some generalizations of the concepts in the previous sections. For this purpose, we can let X and Y be normed linear or locally convex linear Hausdorff spaces and T weakly continuous with $\mathcal{D}(T) = X$ and $\mathcal{R}(T) \subseteq Y$. In this case, T' is weak*-continuous and we can define the states of the operator T (resp. T') following Krishnamurthy (1960) using the weak (resp. weak*) topologies on X and Y (resp. X' and Y').

If we let $\mathcal{W}(X, Y)$ be the set of all weakly continuous operators T with $\mathcal{D}(T) = X$ and $\mathcal{K}(T) \subseteq Y$, then we can define

$D_l = \{T \in \mathcal{W}(X, Y)$; there is an $A \in \mathcal{W}(Y, X)$ with $A \neq 0$ but $T \cdot A = 0\}$

$R_l = \{T \in \mathcal{W}(X, Y)$; there is an $A \in \mathcal{W}(Y, X)$ such that $AT = I\}$

$Z_l = \{T \in \mathcal{W}(X, Y)$; there is a sequence $\{A_n\} \subseteq \mathcal{W}$ such that $TA_n \to 0$ but it is not the case that $A_n \to 0\}$ and similarly for D_r, R_r, and Z_r.

If X and Y are normed spaces then we can consider strong convergence and convergence in the uniform operator topology as well. Thus we define

$$(SZ)_l = \{T \in \mathcal{W}(X, Y);$$

there is a sequence $\{A_n\} \subseteq \mathcal{W}(Y, X)$ such that $TA_n \to 0$ (strongly) but it is not the case that $A_n \to 0$ (strongly)$\}$.

$(UZ)_l = \{T \in \mathcal{W}(X, Y)$; there is a sequence $\{A_n\} \subseteq [Y, X]$ such that $TA_n \to 0$ in $[Y]$ but it is not the case that $A_n \to 0$ in $[Y, X]\}$.

We leave it to the interested reader to continue the investigation into this subject. The proofs in section 2 make it reasonable to expect that many of the results in sections 2 and 3 will hold for the concepts defined in this section.

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