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# ON WEIGHTED FRACTIONAL INTEGRALS

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A new proof, based upon the 'concrete representation' of the adjoint in the 1-dimensional case, of a theorem of Stein and Weiss (1958) on weighted fractional integrals on Euclidean $n$-spaces is given. Our techniques clearly display the fact that the theorem of Stein and Weiss is but a natural successor to the unweighted 1-dimensional result of Hardy and Littlewood (1928). Applications of our method to some other weighted singular and weakly singular integrals are also considered.

## 1. INTRODUCTION

Let $x = (x_1, \ldots, x_n)$ be a general point in Euclidean $n$-space $E^n$ $(E^1 = E)$, $|x| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, and $dx = dx_1 \ldots dx_n$ the usual Euclidean Lebesgue measure on $E^n$. Let $L_{p,na}(E^n)$ and $L_{(p,a)}(E^n)$ $(L_p(E^n) = L_{p,0}(E^n) = L_{(p,0)}(E^n))$ denote, respectively, the spaces of equivalence classes of Lebesgue measurable functions $f$ on $E^n$ for which

$$\|f\|_{p,na} = \left(\int_{E^n} |f(x)|^p \, |x|^{pna} \, dx\right)^{1/p} < \infty$$

and

$$\|f\|_{(p,a)} = \left(\int_{E^n} |f(x)|^p \prod_{i=1}^{n} |x_i|^{pa} \, dx\right)^{1/p} < \infty$$

(usual sup norm if $p = \infty$). Let $C(A, B)$, $A$ and $B$ normed linear spaces, denote the space of linear transformations which are continuous on $A$ into $B$. Let

$$R_d f(u) = \int_{E^n} f(x) \, |u - x|^{d-n} \, dx, \quad 0 < d < n, \quad u \in E^n.$$

*Theorem A* (Hardy-Littlewood 1928) — If $n = 1$, $(1/p) - (1/s) = d$, $1 < p < s < \infty$, and $d < 1$, then $R_d \in C(L_p(E), L_s(E))$.

Theorem (A) was extended to $n$-dimensions by Sobolev (see Cotlar and Ortiz 1957), and the following weighted version of Sobolev's theorem was proved by Stein and Weiss (1958a).

*Theorem B* — If $0 \leqslant a + b \leqslant d/n < 1/p'$, $0 < d$, $1 < p \leqslant q < \infty$, $a < 1/p'$, $b < 1/q$ and $1/q = (1/p) - (d/n) + a + b$, then $R_d \in C(L_{p,na}(E^n), L_{q,nb}(E^n))$.

Various proofs of Theorem B, with varying degrees of difficulty, have appeared over the years (see Coifman 1968, Cotlar and Ortiz 1957, Okikiolu 1971). Our purpose in this note is to give a very elementary proof, using essentially no more than Theorem A, of Theorem B and some related results on singular integral operators. We emphasise that our intention here is not to prove the most general form of Theorem B [we refer the interested reader to Muckenhoupt and Wheeden (1974) and Welland (1975)], but to show that Theorem B is but a natural sequel to Theorem A. We achieve this by showing that if $n = 1$, then $R_d \in C(L_p(E), L_s(E))$, $p$, $s$ and $d$ as in Theorem A, automatically implies that $R_d \in C(L_{p,a}(E), L_{s,a}(E))$ for all $-1/s < a < 1/p'$. A simple argument then shows that $R_d$, for general values of $n$, $\in C(L_{(p,a)}(E^n), L_{(q,b)}(E^n))$, where $p$, $q$, $a$, $b$ and $d$ are as in Theorem B. The proof of Theorem B thereafter remains but a formality.

## 2. Notation

In addition to the notation already introduced, we shall, in the sequel, denote by $W$ a function on $[0, \infty) \to [0, \infty)$ which is increasing and which satisfies the property that

$$W(\,|x + y|\,) \leqslant W(\,|x|\,) + W(\,|y|\,) \qquad \ldots(1)$$

for all $x, y \in E^n$. We write $W(x)$ for $W(\,|x|\,)$. A simple argument shows that if $|x| \leqslant |u|$ then there exists a constant $A \geqslant 1$ such that

$$W(u) \leqslant A(\,|u|\,/\,|x|\,)\,W(x).$$

In the following $L_{p;na}(E^n)$ will denote the space of equivalence classes of Lebesgue measurable functions $f$ on $E^n$ for which

$$\Big\|f\Big\|_{p;na} = \Big(\int_{E^n} \Big|f(x)\Big|^p W(x)^{pna}\,dx\Big)^{1/p} < \infty.$$

We assume, unless otherwise specified, henceforth that $1 \leqslant p, q, s < \infty$. The index conjugate to $p$ will be denoted by $p'$, and given a linear mapping $T$, $T'$ will denote the mapping adjoint to $T$.

The mapping $T_d$ will be defined by

$$T_d f(u) = \int_{E^n} K_d(u - x)\,f(x)\,dx \qquad \ldots(2)$$

where the kernel function $K_d$, assumed hereafter to be positive, satisfies the homogeneity property that

$$K_d(\alpha u - \alpha x) = \alpha^{d-n} K_d(u - x), \quad 0 < d < n, \qquad \ldots(3)$$

for some or all $\alpha > 0$.

$Q_\mu$, $\mu$ some real number, will denote the mapping $Q_\mu f(t) = | t |^{-\mu}(\operatorname{sgn} t)\, f(t^{-1})$; $t \in E$. It is clear that $Q_\mu$ is an isometric isomorphism of $L_p(E)$ onto $L_{p,(\mu-2)/p}(E)$. Let $T \in C(L_p(E), L_q(E))$ be the mapping

$$Tf(y) = \int_{-\infty}^{\infty} K(y, t) f(t)\, dt, \quad y \in E,$$

where

$$K(\alpha y, \alpha t) = | \alpha |^{\mu-1} K(y, t)$$

for all nonzero real $\alpha$. Then it can be shown that $T' = Q_{1-\mu} T Q_{1+\mu}$ whenever $1 < p, q$, and $0 \leqslant \mu < 1/p$ [see Duggal 1974, Lemma (2.9)]. The mapping $T$ above is said to be 0-adjoint if $K(y, t) = AK(t, y)$ for some complex constant $A$ such that $| A | = 1$ (cf. Theorem 3.3 and Definition 3.2 of Duggal 1974); this in fact implies that $T' = AT$, so that $T \in C(Q_{1+\mu} L_p(E), Q_{1-\mu} L_q(E))$ simultaneously.

Throughout this paper $A$, $A_1$, $A_2$ etc. will denote constants, not necessarily the same at different occurrences, which do not depend upon the functions $f$. Whenever the integration extends over all of $E^n$ (or $E^n \times E^n$) the limits will be omitted from the integrals.

## 3. Some Inequalities

In the following we let $g$ be a function in $L_{q'}(E^n)$ such that $\| g \|_{q'} = 1$, and we set $f_0(x) = W(x)^{na} f(x)$.

*Theorem* 3.1 — If $T_{d_1} \in C(L_{p,na}(E^n), L_{s,na}(E^n))$ for all $0 < d_1 < n$,

$$-1/s < a < 1/p' \text{ and } 1/s = (1/p) - (d_1/n),$$

and if $W(x)^{-r} \leqslant AK_{n-r}(x)$ for all $r \geqslant 0$, then $T_d \in C(L_{p;na}(E^n), L_{q;-nb}(E^n))$ for all $0 \leqslant a + b < d/n$, $a < 1/p'$, $b < 1/q$ and $1/q = (1/p) - (d/n) + a + b$.

Proof : We divide the proof of the theorem into two parts, and start by showing in part (i) that if $T$ is the positive kernel operator

$$Tf(u) = \int K(u, x) f(x)\, dx$$

such that $T \in C(L_p(E^n), L_s(E^n))$ and $T \in C(L_{p,na}(E^n), L_{s,na}(E^n))$ for some range of values of $a$, $p$ and $s$, then $T \in C(L_{p;na}(E^n), L_{s;na}(E^n))$ for the same range of values of $a$, $p$ and $s$.

Let $S_1$ and $S_2$ be the disjoint regions

$$S_1 = \{(u, x) : |u| \leqslant |x|\},\ S_2 = \{(u, x) : |x| < |u|\}.$$

(i) To prove that $T \in C(L_{p;na}(E^n), L_{s;na}(E^n))$ it is enough to show that

$$B_i(f, g) = \iint_{S_i} g(u)\, W(u)^{na}\, K(u, x)\, W(x)^{-na} f_0(x)\, dx\, du \quad (i = 1, 2)$$

$$\leqslant A_i \| f_0 \|_p$$

for positive $f$ and $g$. Let $a > 0$. Over $S_1$, $(W(u)/W(x))^{na} \leqslant 1$, so that

$$B_1(f, g) \leqslant \iint_{S_1} g(u)\, K(u, x)\, f_0(x)\, dx\, du \leqslant \iint g(u)\, K(u, x)\, f_0(x)\, dx\, du$$

$$\leqslant A_1 \| f_0 \|_p$$

since $T \in C(L_p(E^n), L_s(E^n))$. Again, over $S_2$, $W(u)/W(x) < A(|u| / |x|)$ so that

$$B_2(f, g) \leqslant A \iint_{S_2} |u|^{na} g(u)\, K(u, x)\, f_0(x)\, |x|^{-na}\, dx\, du$$

$$\leqslant A_2 \| f_0 \|_p$$

since $T \in C(L_{p,na}(E^n), L_{s,na}(E^n))$. In the case in which $a < 0$, we consider $B_1$ and $B_2$ with the regions $S_1$ and $S_2$ interchanged when an argument similar to that above shows that $T \in C(L_{p;na}(E^n), L_{s;na}(E^n))$.

(ii) Once again to prove that $T_d \in C(L_{p;na}(E^n), L_{q;-nb}(E^n))$ it is enough to show that

$$B_i(f, g) = \iint_{S_i} W(u)^{-nb}\, g(u)\, K_d(u - x)\, W(x)^{-na} f_0(x)\, dx\, du \quad (i = 1, 2)$$

$$\leqslant A_i \| f_0 \|_p$$

for positive $f$ and $g$. Over $S_1$, $K_{n-na-nb}(u - x) \geqslant A_5^{-1} W(u - x)^{-na-nb}$

$$\geqslant A_4^{-1} W(2x)^{-na-nb} \geqslant A_3^{-1} W(x)^{-na-nb}, \text{ if}$$

$0 \leqslant a + b$. Thus, if $0 \leqslant a + b$, then

$$B_1(f, g) \leqslant A_3 \iint_{S_1} W(u)^{-nb}\, g(u)\, K(u - x)\, W(x)^{nb} f_0(x)\, dx\, du,$$

where $K(u - x) = K_d(u - x)\, K_{n-na-nb}(u - x)$. Set $d_1 = d - na - nb$,

$$1/p = (1/q) + (d_1/n) = 1/s - a - b + d_1/n.$$

Then $0 < d_1 < n$. Since $T_{d_1} \in C(L_{p,-nb}(E^n), (L_{q,-nb}(E^n))$ for all $1/q = (1/p) - (d_1/n)$ and $-1/p' < b < 1/q$, we have from part (i) that

$$B_1(f, g) \leqslant A_1 \| f_0 \|_p.$$

Again, over $S_2$, $K_{n-na-nb}(u-x) \geqslant A_4^{-1}\ W(u)^{-na-nb}$ if $a+b \geqslant 0$. Thus, if $0 \leqslant a+b$, then

$$B_2(f, g) \leqslant A_4 \iint_{S_2} W(u)^{na}\, g(u)\, K(u-x)\ W(x)^{-na} f_0(x)\, dx\, du.$$

An argument similar to that above shows that

$$B_2(f, g) \leqslant A_2 \parallel f_0 \parallel_p$$

whenever $\quad -1/q < a < 1/p'$ and $1/p = 1/q + d/n - a - b$.

*Remark I* : Let $E^m$ denote the subspace (of $E^n$) $E^m = \{x = (x_1, \ldots, x_n) : x_j = 0$ for $m < j \leqslant n\}$. It is not difficult to see (in fact, proceed exactly as before) that under the same hypotheses on $W$ and $K_d$ as in Theorem (3.1), if

$$T_{d_1} \in C(L_{p,na}(E^n), (L_{s,na}(E^m))$$

for all $0 < d_1 < n$, $-m/(ns) < a < 1/p$ and $m/s = (n/p) - d_1$, then

$$T_d \in C(L_{p,na}(E^n), L_{q,-mb}(E^m))$$

whenever $0 \leqslant na + mb < d - n + m$, $a < 1/p'$, $b < 1/q$ and

$$m/q = (n/p) - d + na + mb.$$

For the purposes of our next result, we let $W(x) = \prod_{i=1}^{n} W(x_i)^{1/n}$ where

$$W(x_i) = W(\mid x_i \mid)$$

satisfies (1) for all $i = 1, \ldots, n$.

*Theorem* 3.2 — Let $K_d(u-x) \leqslant \prod_{i=1}^{n} k^i(u_i - x_i)$, where, for all $i = 1, \ldots, n$,

(i) $k^i(u_i - x_i) = k^i(x_i - u_i)$;

(ii) $k^i(\alpha u_i - \alpha x_i) = \mid \alpha \mid^{(d/n)-1} k^i(u_i - x_i)$ for all non-zero real $\alpha$. Let $V_i$,

$$V_i f(y) = \int_{-\infty}^{\infty} k^i(y-t)\, f(t)\, dt, \quad i = 1, \ldots, n, y \in E,$$

$\in C(L_p(E), L_s(E))$ for all $1 < p, s$ and $1/s = (1/p) - (d/n)$. Then

$$T_d \in C(L_{p;na}(E^n), L_{q;-nb}(E^n))$$

for all $1 < p, q$, $0 \leqslant a + b < d/n$, $a < 1/p'$, $b < 1/q$ and

$$1/q = (1/p) - (d/n) + a + b.$$

If in addition $V_i$ is also continuous on

$$L_{p,\mu}(E) \to L_r(E),\ 1/r = (1/p) - (d/n) - \mu,$$

for some $0 < \mu \leqslant d/n$, then $T_d \in C(L_{p;na}(E^n), L_{q;-nb}(E^n))$ for all $1 < p \leqslant q$, $0 \leqslant a + b \leqslant d/n$, $a < 1/p'$, $b < 1/q$ and $1/q = (1/p) - (d/n) + a + b$.

PROOF: Part of the argument below appears in Duggal (1978); however we include it here for completeness.

We show first of all that $V_i \in C(L_{p,a}(E), L_{s,a}(E))$ for all $-1/s < a < 1/p'$. In view of the remarks made in Section 2, hypotheses (i) and (ii) imply that the mapping $V_i$ is 0-adjoint and that $V_i' = Q_{1-d/n} V_i Q_{1+d/n}$. This implies that

$$V_i \in C(L_{p,1+d/n-(2/p)}(E), L_{s,1-d/n-(2/s)}(E)).$$

Since $1/s = (1/p) - (d/n)$, it follows upon setting $\nu = (1/p') - (1/s)$ that

$$V_i \in C(L_p(E), L_s(E)) \text{ and } C(L_{p,\nu}(E), L_{s,\nu}(E))$$

simultaneously. Now set $a = \nu t = t((1/p') - (1/s))$, $0 \leqslant t \leqslant 1$; then

$$-(1/p) + (d/n) = -1/s < a < 1/p',$$

and it follows from an application of the weighted version of the interpolation theorem (see Stein and Weiss 1958b) that $V_i \in C(L_{p,a}(E), L_{s,a}(E))$.

As before, to prove the theorem it is enough to show that

$$\begin{aligned} B(f, g) &= \iint g(u)\, W(u)^{-nb}\, K_d(u - x)\, W(x)^{na} f_0(x)\, dx\, du \\ &\leqslant \iint g(u) \prod_{i=1}^{n} W(u_i)^{-b} \prod_{i=1}^{n} k^i(u_i - x_i) \prod_{i=1}^{n} W(x_i)^a f_0(x)\, dx\, du \\ &\leqslant A \, \| f_0 \|_p \end{aligned}$$

for positive $f$ and $g$. This will follow from an $n$-times repeated application of the one-dimensional result in so far as we are able to prove that $V_i \in C(L_{p,a}(E), L_{q,-b}(E))$ whenever $a < 1/p'$, $b < 1/q$, $0 \leqslant a + b \leqslant d/n$ and $1/q = (1/p) - (d/n) + a + b$. If $0 \leqslant a + b < d/n$, this follows from Theorem (3.1) since $V_i \in C(L_{p,a}(E), L_{s,a}(E))$ whenever $1/s = (1/p) - (d/n)$ and $-1/s < a < 1/p'$. In the case in which $0 < a + b = d/n$ (the remaining case) we proceed as follows.

As already seen, $V_i \in C(L_{p,\alpha}(E), L_{s,\alpha}(E))$ for all $-(1/p) + (d/n) < \alpha < 1/p'$. If, also, $V_i \in C(L_{p,\mu}(E), L_r(E))$, then we have from the weighted version of the interpolation theorem that $V_i \in C(L_{p,a}(E), L_{q,-b}(E))$, where $a = t\alpha + (1 - t)\,\mu$,

$$b = -t\alpha (0 \leqslant t \leqslant 1)$$

and $1/q = (1/p) - (d/n) + a + b$. Since $0 < \mu \leqslant (1/p) - (1/s)$, it follows that $a < 1/p'$, $b < 1/q$ and $0 \leqslant a + b \leqslant d/n$.

This completes the proof.

The following theorem will be useful in our discussion of weighted singular integrals.

*Theorem* 3.3 — Let

$$T_d^r f(u) = \int K_d(u-x) f(x)\, W(x)^{-r}\, dx, \quad r > 0,\ 0 \leqslant d < n.$$

If $T_1^1 \in C(L_p(E^n), L_p(E^n))$ and $T_r^r \in C(L_p(E^n), L_p(E^n))$, and if $W(x)^r \leqslant K_{n+r}(x)$ for all $r > 0$, then

$$\{\int | \int | 1 - (W(u)/W(x))^r \mid K_0(u-x) f(x)\, dx \mid^p du\}^{1/p} \leqslant A \| f \|_p.$$

PROOF : Set $K(u, x) = | W(x)^r - W(u)^r | W(x)^{-r} K_0(u-x)$, and divide $E^n \times E^n$ into the disjoint regions

$$S_1 = \{(u, x) : W(u) \leqslant 2W(u-x)\};\ S_2 = \{(u, x) : W(u) > 2W(u-x)\}.$$

Over $S_1$, since $r > 0$,

$$W(x)^r \leqslant (W(u-x) + W(u))^r \leqslant 2^r W(u-x)^r + 2^r W(u)^r$$

so that

$$\begin{aligned} W(x)^r - W(u)^r &\leqslant 2^r W(u-x)^r + (2^r - 1)\, W(u)^r \\ &\leqslant 2^r W(u-x)^r + (2^{2r} - 2^r)\, W(u-x)^r \\ &\leqslant A W(u-x)^r. \end{aligned}$$

From the symmetry of $W(u-x)^r$ it follows that

$$| W(x)^r - W(u)^r | \leqslant A W(u-x)^r \leqslant A K_{n+r}(u-x),$$

and so that

$$K(u, x) \leqslant A K_1(u-x)\, W(x)^{-1}.$$

Over $S_2$, $1/2 < W(x)/W(u) < 3/2$, and so

$$\begin{aligned} | W(x)^r - W(u)^r | &= r \mid W(x) - W(u) \mid (W(v)/W(x))^{r-1}\, W(x)^{-1} \\ &\qquad (v \text{ some intermediate point}) \\ &\leqslant A W(u-x)\, W(x)^{r-1}. \end{aligned}$$

Now, if $r \geqslant 1$, then $K(u, x) \leqslant A K_1(u-x)\, W(x)^{-1}$, and if $r < 1$, then

$$\begin{aligned} K(u, x) &\leqslant A K_0(u-x)\, W(u-x)\, W(x)^{-1} \\ &= A K_0(u-x)\, W(u-x)^r\, W(x)^r\, (W(u-x)/W(x))^{r-1} \\ &\leqslant A_1 K_r(u-x)\, W(x)^{-r}. \end{aligned}$$

Thus the proof, in either of the cases above, follows from the hypotheses on $T_r^r$

## 4. Applications

*Theorem* 4.1 — Theorem B holds.

Proof: As in the proof of Theorem (3.2), we consider the cases $a + b < d/n$ and $a + b = d/n$ separately. Let $S_1$ and $S_2$ be the regions

$$S_1 = \{(u, x) : |u| \leqslant 2|x|\} \text{ and } S_2 = \{(u, x) : 2|x| < |u|\}.$$

(i) $a + b < d/n$ : In view of Theorem 3.1, it is enough to show that

$$B_i(f, g) = \iint_{S_i} |u|^{na} g(u) |u - x|^{d-n-na-nb} |x|^{-na} f_0(x)\, dx\, du$$

$$(i = 1, 2; f_0(x) = |x|^{na} f(x), \|g\|_{q'} = 1)$$

$$\leqslant A_i \|f_0\|_p \qquad \ldots(4)$$

for positive $f$ and $g$. Let $a \geqslant 0$. Then

$$B_1(f, g) \leqslant 2^{na} \iint_{S_1} g(u) |u - x|^{d-n-na-nb} f_0(x)\, dx\, du$$

$$\leqslant 2^{na} \iint g(u) \prod_{i=1}^{n} |u_i - x_i|^{(d/n)-1-a-b} f_0(x)\, dx\, du,$$

and

$$B_2(f, g) \leqslant 2^{na} \iint_{S_2} g(u) |u - x|^{d-n-nb} |x|^{-na} f_0(x)\, dx\, du$$

$$\leqslant 2^{na} \iint g(u) \prod_{i=1}^{n} |u_i - x_i|^{(d/n)-n-nb} \prod_{i=1}^{n} |x_i|^{-a} f_0(x)\, dx\, du$$

$$= 2^{na} \iint g(u) \prod_{i=1}^{n} |u_i - x_i|^{d'/n-1} \prod_{i=1}^{n} |x_i|^{-a} f_0(x)\, dx\, du.$$

Since $d < n$ and $a < 1/p'$, $0 < d' < n$. An application of Theorem 3.2, with $b = 0$ and $d' = d$ for $B_2(f, g)$, shows that (4) holds.

The proof for the case in which $a < 0$ follows from the following duality argument.

$$B(f, h) = \iint |u|^{-na} h(u) |u - x|^{d_1-n} f(x)\, dx\, du$$

$$(h \in L_{q'}(E^n), d_1 = d - na - nb)$$

$$= \int f_0(x) [\, |x|^{na} R'_{d_1} (|u|^{na} h(u)) (x)]\, dx$$

$$\leqslant A \|f_0\|_p \|h\|_{q'},$$

if $0 \leqslant -a < 1/q$, i.e. $-1/q < a \leqslant 0$.

(ii) $a + b = d/n$: Let $b < 0$; then, with $f$ and $g$ as in (i),

$$B_1(f, g) = \iint_{S_1} | u |^{-nb} g(u) | u - x |^{d-n} | x |^{-na} f_0(x)\, dx\, du$$

$$\leqslant 2^{-nb} \iint_{S_1} g(u) | u - x |^{d-n} | x |^{-na-nb} f_0(x)\, dx\, du$$

$$\leqslant 2^{-nb} \iint g(u) \prod_{i=1}^{n} | u_i - x_i |^{d/n-1} \prod_{i=1}^{n} | x_i |^{-a-b} f_0(x)\, dx\, du;$$

$$B_2(f, g) = \iint_{S_2} | u |^{-nb} g(u) | u - x |^{d-n} | x |^{-na} f_0(x)\, dx\, du$$

$$\leqslant 2^{na} \iint_{S_2} | u |^{-nb-na} g(u) | u - x |^{d-n+na} | x |^{-na} f_0(x)\, dx\, du$$

$$\leqslant 2^{na} \iint \prod_{i=1}^{n} | u_i |^{-b-a} \prod_{i=1}^{n} | u_i - x_i |^{(d/n)-1+a} \prod_{i=1}^{n} | x_i |^{-a} \times f_0(x)\, dx\, du.$$

Thus the proof in either case follows from Theorem 3.2 since (this is easily verified : see e.g. Duggal 1974, or Okikiolu 1971), for $n = 1$, $R_d \in C(L_{p,\mu}(E), L_r(E))$ whenever $1/r = (1/p) - d + \mu$ and $0 < \mu \leqslant d < 1/p'$.

The proof for the case in which $b > 0$ now follows from a duality argument if $a < 0$ and an argument similar to that above if $a > 0$.

*Remark II*: Simple extension arguments, and the first part of the proof of Theorem 3.1, show that Theorem 4.1 remains valid if one replaces (the weights) $| x |$ by $(1 + | x |)$ or $\sum_{i=1}^{n} | x_i |$ or $\max | x_i |$ or $(\sum_{i=1}^{n} | x_i |^{r})^{1/r}$, $1 \leqslant r < \infty$, or $\prod_{i=1}^{n} (c + | x_i |)$, $c \geqslant 0$.

*Remark III*: Cotlar and Ortiz (1957) have shown that if $1 < p, q$,

$$0 \leqslant na + mb < d - n + m,\ d < n/p,\ a < 1/p',\ b < 1/q$$

and $m/q = (n/p) - d + na + mb$, then $R_d \in C(L_{p,na}(E^n), L_{q,-mb}(E^m))$. A simple proof of this theorem can be given as follows.

With $f$, $g$, $f_0$ and $B_i(f, g)$ as in the proof of Theorem 4.1, it is (in view of Theorem 3.1) enough to show that inequality (4) is satisfied. Let $a > 0$. Clearly,

$$B_1(f, g) \leqslant 2^{na} \iint g(u) | u - x |^{d-n-na-mb} f_0(x)\, dx\, du$$

so that the required inequality follows from an application of Sobolev's theorem (Cotlar and Ortiz 1957, Theorem A). Now set $t = n/r'$, $d - n - na - mb + t = d' - n$ and $0 < a < 1/r' < 1/p'$. Then $0 < d' < n$, and

$$B_2(f, g) \leqslant 2^t \int_{E^m} g(u) \mid u \mid^{na-t} du$$

$$\times \int_{2 \mid x \mid < \mid u \mid} \mid u - x \mid^{d-n-na-mb+t} f_0(x) \mid x \mid^{-na} dx$$

$$\leqslant A_1 \int_{E^m} g(u) \mid u \mid^{na-n/r'} du$$

$$\times \{ \int_{2 \mid x \mid < \mid u \mid} \mid u - x \mid^{d'-n} \mid f_0(x) \mid^r dx\}^{1/r}.$$

$$\times \{ \int_{2 \mid x \mid < \mid u \mid} \mid x \mid^{-nar'} dx\}^{1/r'}$$

$$\leqslant A_1 \int_{E^m} g(u) \, (T_{d'} F(u)^{1/r} du) \qquad (F(x) = f_0(x)^r)$$

$$\leqslant A_1 \parallel g \parallel_{q'} \parallel T_{d'} F \parallel_{q/r}^{1/r}$$

$$\leqslant A \parallel F \parallel_{p/r}^{1/r} \quad \text{(by Sobolev's theorem)}$$

$$= A \parallel f_0 \parallel_p.$$

The proof for the case in which $a < 0$ follows from a duality argument.

Let $0 < s_i$, $\sum_{i=1}^{n} s_i = n$, $V(x) = \sum_{i=1}^{n} s_i \mid x_i \mid^{1/s_i}$, and let

$$Tf(u) = \int f(x) \, V(u - x)^{d-n} \, dx, \qquad 0 < d < n.$$

*Theorem* 4.2 — If $1 < p \leqslant q$, $a < 1/p'$, $b < 1/q$, $0 \leqslant a + b \leqslant d/n < 1/p'$ and $1/q = 1/p - d/n + a + b$, then

$$\parallel Tf(u) \, V(u)^{-nb} \parallel_q \leqslant A \parallel f(x) \, V(x)^{na} \parallel_p$$

whenever the right hand side exists.

PROOF : Since

$$V(u - x) \geqslant A \prod_{i=1}^{n} \mid u_i - x_i \mid^{1/n}$$

a proof of the theorem will follow from Theorems 3.1 and 4.1 once we are able to show that $V(x)^r = W(x)$, $r = \min_{1 \leqslant i \leqslant n} s_i$, is a weight function satisfying inequality (1). This follows since $W$ is a nonnegative measurable function satisfying

$$W(x + y) = \{ \sum_{i=1}^{n} s_i \mid x_i + y_i \mid^{1/s_i}\}^r$$

$$\leqslant \{ \sum_{i=1}^{n} [( \mid t_i x_i + t_i y_i \mid )^{r/s_i}]^{1/r}\}^r \quad (t_i = s_i^{s_i})$$

*(equation continued on p. 586)*

$$\leqslant \{ \sum_{i=1}^{n} [ \mid t_i x_i \mid ^{r/s_i} + \mid t_i y_i \mid ^{r/s_i}]^{1/r} \}^r$$

$$\leqslant \{ \sum_{i=1}^{n} \mid t_i x_i \mid ^{1/s_i} \}^{1/r} + \{ \sum_{i=1}^{n} \mid t_i y_i \mid ^{1/s_i} \}^r$$

$$= W(x) + W(y).$$

*Singular integrals* — We consider now the operator

$$Hf(u) = \int h(u, (u - x)/ \mid u - x \mid ) \mid u - x \mid ^{-n} f(x)\, dx,$$

which, under suitable hypotheses on $h$, is known to be continuous on $L_p(E^n)$, $1 < p$, to itself. In the following we assume that $\sup \mid h(u, (u - x)/ \mid u - x \mid ) \mid < \infty$, and prove, using Theorems 3.3 and 4.1, the following result of Stein (1957) and Timan (1969).

*Theorem* 4.3 — If $-1/p < a < 1/p'$, then $H \in C(L_{p;na}(E^n), L_{p;na}(E^n))$, where $W(x)$ is either $\mid x \mid$ or $\sum_{i=1}^{n} \mid x_i \mid$ or $\prod_{i=1}^{n} \mid x_i \mid ^{1/n}$ or $\max \mid x_i \mid$ or

$$( \sum_{i=1}^{n} \mid x_i \mid ^r)^{1/r},\ 1 \leqslant r < \infty, \text{ or } (1 + \mid x \mid ).$$

PROOF : We prove the theorem for $W(x) = \mid x \mid$ ; the modification required to prove the theorem for the other cases is minor.

Let $F(x) = \mid x \mid ^{na} f(x)$, and let

$$TF(u) = \int \mid 1 - ( \mid u \mid / \mid x \mid )^{na} \mid \ \mid u - x \mid ^{-n} F(x)\, dx.$$

Then to show that $H \in C(L_{p,na}(E^n), L_{p,na}(E^n))$, it is enough to show that

$$T \in C(L_p(E^n), L_p(E^n)).$$

Furthermore, it is enough to consider the case in which $a > 0$ (for the case $a < 0$ follows from a duality argument).

Let $a > 0$; then, because of Theorem 4.1, the hypotheses of Theorem 3.3 are satisfied, and so $T \in C(L_p(E^n), L_p(E^n))$.

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