Indian J. pure appl. Math., **11**(5) : 588-608, May 1980

ELECTROMAGNETIC TENSOR FIELDS IN GENERAL RELATIVITY

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(*Received* 7 *March* 1979; *after revision* 11 *June* 1979)

In this paper, it is shown that the structures of electromagnetic tensor fields in the space time V_4 can be simplified in $V_4 \times R^2$ or $V_4 \times R$. The normality conditions for V_4 have also been obtained.

1. INTRODUCTION

Our operational space is the space-time V_4 of general relativity. g, the metric tensor field may be of index of inertia 2, 0 or 4, that is, of signature $+ + + -$, $+ + - -$ or $+ + + +$, $'k$, the electromagnetic tensor field, is skew-symmetric:

$$'k(X, Y) + 'k(Y, X) = 0 \qquad \text{...(1.1}a\text{)}$$

for arbitrary vector fields X, Y.

Agreement 1.1 — In the above and in what follows the equations containing $X, Y, Z, \ldots$ hold for arbitrary vector fields $X, Y, Z, \ldots$.

We will put

$$g(kX, Y) \overset{def}{=} 'k(X, Y) \qquad \text{...(1.2)}$$

$$\bar{X} \overset{def}{=} kX. \qquad \text{...(1.3)}$$

Then eqn. (1.1a) can be written as

$$g(\bar{X}, Y) + g(X, \bar{Y}) = 0. \qquad \text{...(1.1}b\text{)}$$

The eigenvalues of k are given by

$$\text{(a)} \quad | k - \lambda I_4 | = 0, \qquad \text{(b)} \quad \overset{4}{\lambda} + 2K\overset{2}{\lambda} + | k | = 0 \qquad \text{...(1.4)}$$

where

$$\text{(a)} \quad 4K \overset{def}{=} -tr k^2, \qquad \text{(b)} \quad | k | = \det (k). \qquad \text{...(1.5)}$$

The electromagnetic tensor field $'k$ is said to be of the

(a) first class if $K \, | k | \neq 0$, (b) second class if $K \neq 0, \; | k | = 0$,

(c) third class if $K = 0, \; | k | = 0, \; k^2 \neq 0$, (d) fourth class if $k^2 = 0$.

For the index of inertia 2 of g, the fourth class does not exist. The first two classes of $'k$ belong to non-null electromagnetic fields, whereas third and fourth classes belong to null electromagnetic fields (Hlavatý 1957).

In view of (1.4) and the classification given above the characteristic equations for k are as follows :

First class

$$\text{(a)}\quad k^4 + 2K\,k^2 + |\,k\,|\,I_4 = 0, \quad \text{(b)}\quad D \overset{def}{=} K^2 - |\,k\,| \neq 0, \qquad \text{...(1.6)}$$

$$\text{(a)}\quad k^2 + K\,I_4 = 0, \quad \text{(b)}\quad D = 0. \qquad \text{...(1.7)}$$

Second class

$$k^3 + 2K\,k = 0. \qquad \text{...(1.8)}$$

Third class

$$k^3 = 0. \qquad \text{...(1.9)}$$

Fourth class

$$k^2 = 0. \qquad \text{...(1.10)}$$

Let $\underset{1}{\lambda}, \underset{2}{\lambda}, \underset{3}{\lambda}, \underset{4}{\lambda}$ be the eigenvalues of k. Then in the first class of k, when $D \neq 0$, we have

$$\left.\begin{aligned} &\text{(a)}\quad \underset{1}{\lambda} = -\underset{2}{\lambda} = i(\sqrt{D} + K)^{1/2} = Mi \sin\sigma \\ &\text{(b)}\quad \underset{3}{\lambda} = -\underset{4}{\lambda} = (\sqrt{D} - K)^{1/2} = M \cos\sigma \end{aligned}\right\} \qquad \text{...(1.11)}$$

where

$$\text{(a)}\quad M^2 = 2\sqrt{D}, \qquad \text{(b)}\quad \tan\sigma = (\sqrt{D} + K)^{1/2}/(\sqrt{D} - K)^{1/2}. \qquad \text{...(1.12)}$$

When $D = 0$,

$$\underset{1}{\lambda} = -\underset{2}{\lambda} = \underset{3}{\lambda} = -\underset{4}{\lambda} = -K. \qquad \text{...(1.13)}$$

In the second class of $'k$, we have

$$\text{(a)}\quad \underset{1}{\lambda} = -\underset{2}{\lambda} = \sqrt{-2K}, \quad \text{(b)}\quad \underset{3}{\lambda} = -\underset{4}{\lambda} = 0,\ K > 0 \qquad \text{...(1.14)}$$

$$\text{(a)}\quad \underset{1}{\lambda} = -\underset{2}{\lambda} = 0, \quad \text{(b)}\quad \underset{3}{\lambda} = -\underset{4}{\lambda} = \sqrt{-2K},\ K < 0. \qquad \text{...(1.15)}$$

In the third and fourth classes of $'k$, all the eigenvalues vanish.

2. Structure in V_4

We will first consider the first class of $'k$ when $D \neq 0$.

Let $\underset{1}{P}, \underset{2}{P}, \underset{1}{R}, \underset{2}{R}$ be the eigen vectors of k corresponding to the distinct eigen-values $\underset{1}{\lambda}, \underset{2}{\lambda}, \underset{3}{\lambda}, \underset{4}{\lambda}$. Then the set $\{\underset{1}{P}, \underset{2}{P}, \underset{1}{R}, \underset{2}{R}\}$ is L.I. Let $\{\overset{1}{p}, \overset{2}{p}, \overset{1}{r}, \overset{2}{r}\}$, be the set inverse to $\underset{1}{P}, \underset{2}{P}, \underset{1}{R}, \underset{2}{R}$. We then have

$$I_4 = \overset{x}{p} \otimes \underset{x}{P} + \overset{x}{r} \otimes \underset{x}{R}, \; x = 1, 2, \qquad \text{...(2.1)}$$

$$k^2 = -M^2 \sin^2 \sigma \overset{x}{p} \otimes \underset{x}{P} + M^2 \cos^2 \sigma \overset{x}{r} \otimes \underset{x}{R}. \qquad \text{...(2.2)}$$

In view of (2.1), eqn. (2.2) can be written as

$$k^2 + M^2 \sin^2 \sigma I_4 = M^2 \overset{x}{r} \otimes \underset{x}{\bar{R}} \qquad \text{...(2.3a)}$$

where

$$\left.\begin{array}{l} \underset{x}{\bar{R}} = r_x^y \underset{y}{R}, \quad \overset{x}{r}(\bar{X}) = r_y^x \overset{y}{r}(X) \\ r_1^1 = -r_2^2 = M \cos \sigma, \; r_2^1 = r_1^2 = 0 \end{array}\right\} \qquad \text{...(2.3b)}$$

or

$$k^2 - M^2 \cos^2\sigma \, I_4 + M^2 \overset{x}{p} \otimes \underset{x}{\bar{P}} = 0 \qquad \text{...(2.4a)}$$

where

$$\left.\begin{array}{l} \underset{x}{\bar{P}} = p_x^y \underset{y}{P}, \quad \overset{x}{p}(\bar{X}) = p_y^x \overset{y}{p}(X) \\ p_1^1 = -p_2^2 = Mi \sin \sigma, \; p_2^1 = p_1^2 = 0. \end{array}\right\} \qquad \text{...(2.4b)}$$

Since $\{\underset{x}{P}, \underset{x}{R}\}$ are null vectors, we can put, without loss of generality

$$g = \overset{1}{p} \otimes \overset{2}{p} + \overset{2}{p} \otimes \overset{1}{p} + \overset{1}{r} \otimes \overset{2}{r} + \overset{2}{r} \otimes \overset{1}{r}. \qquad \text{...(2.5)}$$

Consequently

$$g(\bar{X}, \bar{Y}) = M^2 \sin^2 \sigma \, g(X, Y) - M^2 \{\overset{1}{r}(X) \overset{2}{r}(Y) + \overset{2}{r}(X) \overset{1}{r}(Y)\}. \qquad \text{...(2.6)}$$

or

$$g(\bar{X}, \bar{Y}) = -M^2 \cos^2\sigma \, g(X, Y) + M^2 \{\overset{1}{p}(X) \overset{2}{p}(Y) + \overset{2}{p}(X) \overset{1}{p}(Y)\}. \qquad \text{...(2.7)}$$

From the above, we see that for the electromagnetic tensor field of the first class when $D \neq 0$, we have the structure $\{k, M, \; , \overset{x}{r}, \underset{x}{R}, r^{y}_{x}, g\}$ given by (2.3) and (2.6) or the structure $\{k, M, \; , \overset{x}{p}, \underset{x}{P}, p^{y}_{x}, g\}$ given by (2.4) and (2.7).

Let us now make the following transformations :

$$\text{(a)} \quad \sqrt{2}\,\underset{1}{P'} \overset{def}{=} \underset{1}{P} + \underset{2}{P}, \qquad \text{(b)} \quad \sqrt{2}\, i\underset{2}{P'} \overset{def}{=} \underset{1}{P} - \underset{2}{P},$$

$$\text{(c)} \quad \sqrt{2}\,\underset{1}{R'} \overset{def}{=} \underset{1}{R} + \underset{2}{R}, \qquad \text{(d)} \quad \sqrt{2}\, i\underset{2}{R'} \overset{def}{=} \underset{2}{R} - \underset{1}{R}, \qquad \text{...(2.8)}$$

$$\text{(a)} \quad \sqrt{2}\,\overset{1}{p'} \overset{def}{=} \overset{1}{p} + \overset{2}{p}, \qquad \text{(b)} \quad \sqrt{2}\, i\overset{2}{p} \overset{def}{=} \overset{2}{p} - \overset{1}{p},$$

$$\text{(c)} \quad \sqrt{2}\,\overset{1}{r'} \overset{def}{=} \overset{1}{r} + \overset{2}{r}, \qquad \text{(d)} \quad \sqrt{2}\, i\overset{2}{r'} \overset{def}{=} \overset{1}{r} - \overset{2}{r}. \qquad \text{...(2.9)}$$

Then it is easy to see that $\{\underset{x}{P'}, \underset{x}{R'}\}$ are L.I. and $\overset{x}{p'}(\underset{y}{P'}) = \overset{x}{r'}(\underset{y}{R'}) = \delta^{x}_{y}$. Equations (2.3) and (2.6) assume the forms

$$\text{(a)} \quad k^2 + M^2 \sin^2 \sigma I_4 = M^2 \overset{x}{r'} \otimes \underset{x}{R'},$$

$$\text{(b)} \quad \overline{\underset{x}{R'}} = \theta^{y}_{x} \underset{y}{R'},\ \theta^{2}_{1} = Mi \cos\sigma,\ \theta^{y}_{x} + \theta^{x}_{y} = 0,\ \overset{x}{r}(\bar{X}) = \theta^{x}_{y} \overset{y}{r}(X),$$

$$\text{(c)} \quad g(\bar{X}, \bar{Y}) = M^2 \sin^2 \sigma\, g(X, Y) - M^2 \overset{x}{r}(X)\, \overset{x}{r}(Y). \qquad \text{...(2.10)}$$

Similarly the eqns. (2.4) and (2.7) assume the forms

$$\text{(a)} \quad k^2 - M^2 \cos^2\sigma\, I_4 + M^2 \overset{x}{p'} \otimes \underset{x}{P'} = 0,$$

$$\text{(b)} \quad \overline{\underset{x}{P'}} = \varphi^{y}_{x} \underset{y}{P'},\ \varphi^{2}_{1} = -M \sin\sigma,\ \varphi^{y}_{x} + \varphi^{x}_{y} = 0,\ \overset{x}{p'} = \varphi^{x}_{y} \overset{y}{p'},$$

$$\text{(c)} \quad g(\bar{X}, \bar{Y}) = - M^2 \cos^2 \sigma\, g(X, Y) + M^2 \overset{x}{p'}(X)\, \overset{x}{p'}(Y). \qquad \text{...(2.11)}$$

Thus for the electromagnetic tensor field of the first class when $D \neq 0$, we also have the structure $\{k, M, \sigma, \overset{x}{r'}, \underset{x}{R'}, \theta^{y}_{x}, g\}$ given by (2.10) or the structure $\{k, M, \sigma, \overset{x}{p'}, \underset{x}{P'}, \varphi^{y}_{x}, g\}$ given by (2.11).

As is clear from (1.7), for the electromagnetic tensor field of the first class when $D = 0$, we have the structure $\{k, g\}$ given by

$$k^2 + K I_4 = 0,\ g(\bar{X}, \bar{Y}) = K g(X, Y).$$

We will now consider the second class of $'k$. The eigenvalues of k are given by (1.14) and (1.15). Though there is a pencil of eigen vectors corresponding to the 0 eigenvalue, we can still find linearly independent set of eigen vectors $\{\underset{x}{P}, \underset{x}{R}\}$ and their dual set $\{\overset{x}{p}, \overset{x}{r}\}$ such that (2.1) is satisfied. We have, when $K > 0$

$$k^2 + 2K\, I_4 = 2K\, \overset{x}{r} \otimes \underset{x}{R}, \qquad \text{(b)} \quad \overline{\underset{x}{R}} = 0, \qquad \text{(c)} \quad \overset{x}{r}(\bar{X}) = 0. \qquad \text{...(2.12)}$$

$$g(\bar{X}, \bar{Y}) = 2K\, g(X, Y) - 2K\, \{\overset{1}{r}(X)\, \overset{2}{r}(Y) + \overset{1}{r}(Y)\, \overset{2}{r}(X)\}. \qquad \text{...(2.13)}$$

When $K < 0$, we have

$$\text{(a)} \quad k^2 + 2K\, I_4 = 2K\, \overset{x}{p} \otimes \underset{x}{P}, \qquad \text{(b)} \quad \overline{\underset{x}{P}} = 0, \qquad \text{(c)} \quad \overset{x}{p}(\bar{X}) = 0 \qquad \text{...(2.14)}$$

$$g(\bar{X}, \bar{Y}) = 2Kg(X, Y) - 2K\, \{\overset{1}{p}(X)\, \overset{2}{p}(Y) + \overset{2}{p}(X)\, \overset{1}{p}(Y)\}, \qquad \text{...(2.15)}$$

Thus for the electromagnetic tensor field of the second class we have the structure $\{k, K, \overset{x}{r}, \underset{x}{R}, g\}$ given by (2.12) and (2.13) when $K > 0$ and the structure $\{k, K, \overset{x}{p}, \underset{x}{P}, g\}$ given by (2.14) and (2.15) when $K < 0$.

We will now make the transformations (2.8) and (2.9). The eqns. (2.12), (2.13), (2.14) and (2.15) reduce to the following :

When $K > 0$,

$$\text{(a)} \quad k^2 + 2K\, I_4 = 2K\, \overset{x}{r}{}' \otimes \underset{x}{R}{}', \qquad \text{(b)} \quad \overline{\underset{x}{R}{}'} = 0, \qquad \text{(c)} \quad \overset{x}{r}{}'(\bar{X}) = 0, \qquad \text{...(2.16)}$$

$$g(\bar{X}, \bar{Y}) = 2K\, g(X, Y) - 2K\, \overset{x}{r}{}'(X)\, \overset{x}{r}{}'(Y). \qquad \text{...(2.17)}$$

When $K < 0$,

$$\text{(a)} \quad k^2 + 2K\, I_4 = 2K\, \overset{x}{p}{}' \otimes \underset{x}{P}{}', \qquad \text{(b)} \quad \overline{\underset{x}{P}{}'} = 0, \qquad \text{(c)} \quad \overset{x}{p}{}'(\bar{X}) = 0, \qquad \text{...(2.18)}$$

$$g(\bar{X}, \bar{Y}) = 2K\, g(X, Y) - 2K\, \overset{x}{p}{}'(X)\, \overset{x}{p}{}'(Y). \qquad \text{...(2.19)}$$

Therefore, for the electromagnetic tensor field of the second class, we have the structure $\{k, K, \overset{x}{r}{}', \underset{x}{R}{}', g\}$ given by (2.16) and (2.17) when $K > 0$ and the structure $\{k, K, \overset{x}{p}{}', \underset{x}{P}{}', g\}$ given by (2.18) and (2.19) when $K < 0$.

For the third class of $'k$, all the eigenvalues vanish. Therefore there is a pencil of (null) eigen vectors corresponding to the quadruple eigenvalue 0. Let $\underset{2}{R}$ be any eigen vector. We can still find a set of linearly independent vectors $\{\underset{x}{P}, \underset{x}{R}\}$ and their dual set $\{\overset{x}{p}, \overset{x}{r}\}$ such that (Mishra 1976) (2.1) and

$$k = \overset{1}{r} \otimes T + t \otimes \underset{2}{R} \qquad \ldots(2.20)$$

where

$$\text{(a)} \quad \sqrt{2}\, T \overset{def}{=} \underset{1}{P} + \underset{2}{P}, \qquad \text{(b)} \quad \sqrt{2}\, t \overset{def}{=} \overset{1}{p} + \overset{2}{p} \qquad \ldots(2.21)$$

are satisfied. Consequently

$$\text{(a)} \quad k^2 = \overset{1}{r} \otimes \underset{2}{R}, \qquad \text{(b)} \quad \overline{\underset{2}{R}} = 0,$$

$$\text{(c)} \quad \overset{1}{r}(\bar{X}) = 0, \qquad \text{(d)} \quad \overset{1}{r}(X) = g(X, \underset{2}{R}), \qquad \ldots(2.22)$$

$$g(\bar{X}, \bar{Y}) = \overset{1}{r}(X)\, \overset{1}{r}(Y). \qquad \ldots(2.23)$$

Thus in the electromagnetic tensor field of the third class, we have the structure $\{k, \overset{1}{r}, \underset{2}{R}, g\}$ given by (2.22) and (2.23).

In the electromagnetic tensor field of the fourth class, we have the structure $\{k, g\}$ given by

$$\text{(a)} \quad k^2 = 0, \qquad \text{(b)} \quad g(\bar{X}, \bar{Y}) = 0. \qquad \ldots(2.24)$$

This structure is called an almost tangent metric structure (Eliopolous 1965).

3. $V_4 \times R^2$ AND $V_4 \times R$

Let us now consider the product space $V_4 \times R^2$, where R^2 is two dimensional Euclidean space with a coordinate system $(\overset{1}{x}, \overset{2}{x})$ and V_4 is the space-time with electromagnetic tensor field of the first class when $D \neq 0$. Let us put

$$\underset{1}{T} \overset{def}{=} \frac{\partial}{\partial \overset{1}{x}}, \quad \underset{2}{T} \overset{def}{=} \frac{\partial}{\partial \overset{2}{x}}. \qquad \ldots(3.1)$$

A tangent vector $\tilde{X}$ of $V_4 \times R^2$ has a direct sum decomposition

$$\tilde{X} = X + a^x \underset{x}{T}. \qquad \ldots(3.2)$$

Let us define a vector valued linear function F in $V_4 \times R^2$ as

$$F\tilde{X} \overset{def}{=} \bar{X} - M a^x \underset{x}{R'} + \{M \overset{y}{r'}(X) - \overset{x}{a}\, \theta^{y}_{x}\} \underset{y}{T}, \qquad \ldots(3.3a)$$

where

$$\theta_1^2 = Mi \cos \sigma, \ \theta_x^y + \theta_y^x = 0. \qquad ...(3.3b)$$

Pre-multiplying (3.3*a*) by *F*, using (3.3*a*, *b*) and (2.10*a*), we easily get

$$F^2\tilde{X} + M^2 \sin^2 \sigma \tilde{X} = 0. \qquad ...(3.4)$$

The above structure is a π-structure (Legrand 1966). We, therefore, have the following theorem :

Theorem 3.1 — A space-time V_4, with electromagnetic tensor field of the first class when $D \neq 0$ can always be considered as a π-structure manifold $V_4 \times R^2$, with the π-structure $\{F, -M \sin \sigma\}$ where F is given by (3.3).

We will now define the vector valued linear function F in $V_4 \times R^2$ as

$$F\tilde{X} \stackrel{def}{=} \bar{X} - Mi\overset{x}{a}\underset{x}{P'} + \{Mi\overset{y}{p'}(X) - a^x \theta_x^y\} \underset{y}{T} \qquad ...(3.5a)$$

where

$$\theta_1^2 = -M \sin \sigma, \ \theta_x^y + \theta_y^x = 0. \qquad ...(3.5b)$$

Premultiplying (3.5*a*) by *F*, using (3.5*a*, *b*) and (2.11*a*), we obtain

$$F^2\tilde{X} - M^2 \cos^2 \sigma \tilde{X} = 0. \qquad ...(3.6)$$

This is again a π-structure. We, therefore, have the following theorem :

Theorem 3.2 — A space-time V_4 with electromagnetic tensor field of the first class when $D \neq 0$, can always be considered as a π-structure manifold $V_4 \times R^2$ with the π-structure $\{F, M^2 \cos^2 \sigma\}$, where F is given by (3.5).

For the electromagnetic tensor field of the first class when $D = 0$ the structure k is given by

$$k^2 + K I_4 = 0. \qquad ...(3.7)$$

Thus $\{k, -K\}$ is a π-structure in V_4. In $V_4 \times R^2$, the structure is not simplified. However, we are giving the result without giving details.

Theorem 3.3 — A space-time V_4 with electromagnetic tensor field of the first class when $D = 0$, can also be considered as a π-structure manifold $V_4 \times R^2$ with the π-structure $\{F, -K\}$ given by

$$\text{(a)} \ \ F\tilde{X} \stackrel{def}{=} \bar{X} - a^x\theta_x^y \underset{y}{T}, \qquad \text{(b)} \ \ \theta_1^2 = K, \ \ \theta_x^y + \theta_y^x = 0. \qquad ...(3.8)$$

We will now consider the space-time V_4 with electromagnetic tensor field of the second class. The tensor field F will be defined as follows :

(a) $$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, a^{\alpha} \underset{x}{R} + \sqrt{2K}\, \overset{x}{r}(X) \underset{x}{T}, \quad K > 0 \qquad \text{...(3.9a)}$$

(a) $$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, a^{\alpha} \underset{x}{P} + \sqrt{2K}\, \overset{x}{p}(X) \underset{x}{T}, \quad K < 0. \qquad \text{...(3.10a)}$$

Pre-multiplying (3.9*a*) and (3.10*a*) by F, using these equations and substituting from (2.12) and (2.14) respectively, we get in both the cases

$$F^2\tilde{X} = -2K\,\tilde{X}. \qquad \text{...(3.11)}$$

Similarly if F is defined as

$$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, a^{\alpha} \underset{x}{R'} + \sqrt{2K}\, \overset{x}{r'}(X) \underset{x}{T}, \quad K > 0 \qquad \text{...(3.9b)}$$

$$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, a^{\alpha} \underset{x}{P'} + \sqrt{2K}\, \overset{x}{p'}(X) \underset{x}{T}, \quad K < 0 \qquad \text{...(3.10b)}$$

we get (3.11). Hence, we have the following theorem :

Theorem 3.4 — A space-time V_4 with electromagnetic tensor field of the second class can always be considered as a π-structure manifold $V_4 \times R^2$ with the π-structure $\{F, -2K\}$, where F is given by (3.9) and (3.10).

We continue the study of electromagnetic tensor field of the second class and define F as follows :

(a) $$F\tilde{X} \overset{def}{=} X - \sqrt{-2K}\, a^{\alpha} \underset{x}{P} + \{\sqrt{-2K}\, \overset{y}{p}(X) - a^{\alpha}\theta^{y}_{\alpha}\} \underset{y}{T},$$

(b) $$\theta^1_1 = -\,\theta^2_2 = \sqrt{-2K},\ \theta^2_1 = \theta^1_2 = 0, \qquad \text{...(3.12)}$$

when $K > 0$, and

(a) $$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, a^{\alpha} \underset{x}{R} + \{\sqrt{-2K}\, \overset{y}{r}(X) + a^{\alpha}\theta^{y}_{\alpha}\} \underset{y}{T},$$

(b) $$\theta^1_1 = -\,\theta^2_2 = +\sqrt{-2K},\ \theta^2_1 = \theta^1_2 = 0 \qquad \text{...(3.13)}$$

when $K < 0$. Pre-multiplying (3.12*a*) by F, using (3.11), $\theta^2_1 = \theta^1_2 = 0$, (2.12*a*) and $\underset{x}{\bar{P}} = \theta^{y}_{\alpha} \underset{y}{P}$, $\overset{x}{p}(\bar{X}) = \theta^{\alpha}_{y} \overset{y}{p}(X)$, we obtain

$$F^2\tilde{X} = 0. \qquad \text{...(3.14)}$$

Similarly pre-multiplying (3.13*a*) by F, using (3.13) and

$$\underset{x}{\bar{R}} + \theta^{y}_{\alpha} \underset{y}{R} = 0, \quad \overset{x}{r}(\bar{X}) + \theta^{\alpha}_{y} \overset{y}{r}(X) = 0$$

we again obtain (3.14). Hence, we have the following theorem :

Theorem 3.5 — A space-time V_4 with electromagnetic tensor field of the second class can always be considered as an almost tangent manifold with the almost tangent structure $\{F\}$, where F is given by (3.12) and (3.13).

Finally we define F as follows :

$$\text{(a)} \quad F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, a^x \underset{x}{P'} + \{\sqrt{-2K}\, \overset{y}{p'}(X) - a^x \theta^y_x\} \underset{y}{T},$$

$$\text{(b)} \quad \theta^2_1 = -\theta^1_2 = \sqrt{2K}, \quad \theta^1_1 = \theta^2_2 = 0 \qquad \text{...(3.15)}$$

where $K > 0$ and

$$\text{(a)} \quad F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, a^x \underset{x}{R'} + \{\sqrt{-2K}\, \overset{y}{r'}(X) + a^x \theta^y_x\} \underset{y}{T},$$

$$\text{(b)} \quad \theta^2_1 = -\sqrt{2K}, \theta^y_x + \theta^x_y = 0 \qquad \text{...(3.16)}$$

where $K < 0$. As before, we can prove the following theorem :

Theorem 3.6 — A space-time V_4 with electromagnetic tensor field of the second class can always be considered as an almost tangent manifold with almost tangent structure $\{F\}$, where F is given by (3.15) and (3.16).

We will now take up the cases of null electromagnetic fields. We will first consider V_4 with electromagnetic tensor field of the third class of $'k$ whose structure is given by (2.14).

We consider the product space $V_4 \times R$, where R is the real line. Let t be a unit vector in R. Then a tangent vector $\tilde{X}$ of $V_4 \times R$ has a direct sum decomposition

$$\tilde{X} = X + at \qquad \text{...(3.17)}$$

where a is a real number. Let us define a tensor field F of the type (1, 1) in $V_4 \times R$ as

$$F\tilde{X} \overset{def}{=} \bar{X} + a\underset{2}{R} - \overset{1}{r}(X)\, t. \qquad \text{...(3.18)}$$

Then

$$\text{(a)} \quad FX = \bar{X} - \overset{1}{r}(X)\, t, \qquad \text{(b)} \quad Ft = R_2. \qquad \text{...(3.19)}$$

Pre-multiplying (3.18) by F, using (3.18) and (2.22) we obtain $F^2\tilde{X} = 0$. Hence we have the following theorem :

Theorem 3.7 — A space-time V_4 with electromagnetic tensor field of the third class can always be considered as an almost tangent manifold $V_4 \times R$ with the almost tangent structure $\{F\}$ where F is given by (3.18).

For the fourth class of $'k$, we define F in $V_4 \times R$ as

$$F\tilde{X} \overset{def}{=} \bar{X}. \qquad \ldots(3.20)$$

Then $F^2 = 0$. But this result is trivial.

We will now recapitulate the above discussion in tabular form (Table I).

It is seen from the above discussion and tables that in the first three classes of $'k$ (excepting the first class of $'k$ when $D = 0$) the structures of $V_4 \times R^2$ and $V_4 \times R$ in the cases of non-null and null electromagnetic fields respectively given in the last column of the tables are simpler than the structures of V_4 given in the second column of the table. The structures of $V_4 \times R^2$ or $V_4 \times R$ are known structures viz. π-structures or almost tangent structures whereas structures of V_4 are complicated structures not studied so far, in general. Therefore, it is easier to study the structure of $V_4 \times R^2$ and $V_4 \times R$. It is our conjecture that a study of the structures of $V_4 \times R^2$ and $V_4 \times R$ as defined above, would give information about V_4.

4. Normality of V_4 in the First Class of $'k$

We have seen above that in the first class of $'k$ when $D \neq 0$, the manifold $V_4 \times R^2$ is a π-structure manifold with the π-structure $\{F, -M^2 \sin^2 \sigma\}$ or $\{F, M^2 \cos^2 \sigma\}$ given by

$$\text{(a)} \quad F^2 + M^2 \sin^2 \sigma\, I_6 = 0 \quad \text{or} \quad \text{(b)} \quad F^2 - M^2 \cos^2 \sigma\, I_6 = 0. \qquad \ldots(4.1)$$

In the case of (4.1a), F is defined by

$$\text{(a)} \quad F\tilde{X} \overset{def}{=} \bar{X} - Ma^x \underset{x}{R'} + \{M\overset{y}{r'}(X) - a^x \theta^y_x\} \underset{y}{T},$$

$$\text{(b)} \quad \theta^2_1 = Mi \cos \sigma, \qquad \text{(c)} \quad \theta^y_x + \theta^x_y = 0 \qquad \ldots(4.2)$$

whereas in the case of (4.1b), F is defined by

$$\text{(a)} \quad F\tilde{X} \overset{def}{=} \bar{X} - Mia^x \underset{x}{P'} + \{Mi\overset{y}{p'}(X) - a^x\varphi^y_x\} \underset{y}{T},$$

$$\text{(b)} \quad \varphi^2_1 = -M \sin \sigma, \qquad \text{(c)} \quad \varphi^y_x + \varphi^x_y = 0. \qquad \ldots(4.3)$$

It is well known that π-structures are integrable if and only if Nijenhuis tensors vanish.

We will first consider the π-structure $\{F, -M^2 \sin^2 \sigma\}$ of the manifold $V_4 \times R^2$. Let its Nijenhuis tensor be $[F, F]$. Then

$$[F, F](\tilde{X}, \tilde{Y}) = [F\tilde{X}, F\tilde{Y}] + F^2[\tilde{X}, \tilde{Y}] - F[F\tilde{X}, \tilde{Y}] - F[\tilde{X}, F\tilde{Y}]. \qquad \ldots(4.4)$$

For any tangent vector X of V_4, we have from (4.2),

$$\text{(a)} \quad FX = \bar{X} + M\overset{y}{r}'(X) \underset{y}{T}, \qquad \text{(b)} \quad F\underset{x}{T} = -M\underset{x}{R}' - \theta^{y}_{x} \underset{y}{T},$$

$$\text{(c)} \quad \theta^{2}_{1} = Mi \cos \sigma, \qquad \text{(d)} \quad \theta^{y}_{x} + \theta^{x}_{y} = 0. \qquad \ldots(4.5)$$

If X, Y are tangent vectors of V_4, we have the direct sum decomposition

$$[F, F](X, Y) = \underset{0}{N}(X, Y) + \overset{x}{N}(X, Y) \underset{x}{T} \qquad \ldots(4.6)$$

where

$$\underset{0}{N}(X, Y) = [k, k](X, Y) + (dM\overset{x}{r}')(X, Y) M\underset{x}{R}' \qquad \ldots(4.7a)$$

$$\overset{x}{N}(X, Y) = (dM\overset{x}{r}')(\bar{X}, Y) + (dM\overset{x}{r}')(X, \bar{Y}) - M\overset{y}{r}'(Y)(X\theta^{x}_{y})$$

$$+ M\overset{y}{r}'(X)(Y\theta^{x}_{y}). \qquad \ldots(4.7b)$$

From the above, it is clear that if $[F, F] = 0$, then $\underset{0}{N}$ and $\overset{x}{N}$ both vanish. When $\underset{0}{N}$ vanishes, we call V_4 normal space-time.

We now consider the π-structure $\{F, M^2 \cos^2 \sigma\}$. From (4.3), we have

$$\text{(a)} \quad FX = \bar{X} + Mi\overset{x}{p}'(X) \underset{x}{T}, \qquad \text{(b)} \quad F\underset{x}{T} = -Mi\underset{x}{P}' - \theta^{y}_{x} \underset{y}{T},$$

$$\text{(c)} \quad \theta^{2}_{1} = -M \sin \sigma, \qquad \text{(d)} \quad \theta^{y}_{x} + \theta^{x}_{y} = 0. \qquad \ldots(4.8)$$

In this case

$$[F, F](X, Y) = \underset{0}{N}'(X, Y) + \overset{x}{N}'(X, Y) \underset{x}{T} \qquad \ldots(4.9)$$

where

$$\text{(a)} \quad \underset{0}{N}'(X, Y) = [k, k](X, Y) - (dM\overset{x}{p}')(X, Y) M\underset{x}{P}',$$

$$\text{(b)} \quad \overset{x}{N}'(X, Y) = i\{(dM\overset{x}{p}')(\bar{X}, Y) + (dM\overset{x}{p}')(X, \bar{Y})$$

$$- M\overset{y}{p}'(Y)(X\theta^{x}_{y}) + M\overset{y}{p}'(X)(Y\theta^{x}_{y})\}. \qquad \ldots(4.10)$$

From the above discussions, we have the following theorem :

Theorem 4.1 — The condition that the space-time V_4 with the electromagnetic tensor field of the first class when $D \neq 0$ be normal is $\underset{0}{N} = 0$, when the structure

TABLE I
Non-null Electromagnetic Field

Class		Structure in V_4	Structure in $V_4 \times R^2$	F is defined by
I	$D \neq 0$	$k^2 + M^2 \sin^2 \sigma\, I_4 = M^2 \overset{x}{r} \otimes \underset{x}{R},\ x$ $= 1, 2,\ \underset{x}{\bar{R}} = r^{y}_{x} \underset{y}{R},\ r^1_1 = -r^2_2$ $= M \cos \sigma,\ r^2_1 = r^1_2 = 0,$ or $k^2 + M^2 \sin^2 \sigma\, I_4 = M^2 \overset{x}{r'} \otimes \underset{x}{R'},$ $\underset{x}{\bar{R}'} = \theta^{y}_{x} \underset{y}{R'},\ \theta^2_1 = Mi \cos \sigma,$ $\theta^{y}_{y} + \theta^{x}_{y} = 0.$	$F^2 + M^2 \sin^2 \sigma I_6 = 0$	$\tilde{X} = X + a^x \underset{x}{T}$ $F\tilde{X} \overset{def}{=} \bar{X} - ma^x \underset{x}{R'} + \{M\overset{y}{r'}(X) - a^x\theta^{y}_{x}\} \underset{y}{T}$
		$k^2 - M^2 \cos^2 \sigma I_4 + M^2 \overset{x}{p} \otimes \underset{x}{P} = 0,$ $\underset{x}{\bar{P}} = p^{y}_{x} \underset{y}{P},\ p^1_1 = -p^2_2 = Mi \sin \sigma,$ $p^2_1 = p^1_2 = 0;$ or $k^2 - M^2 \cos^2 \sigma I_4 + M^2 \overset{x}{p'} \otimes \underset{x}{P'} = 0$ $\underset{x}{\bar{P}'} = \varphi^{y}_{x} \underset{y}{P'},\ \varphi^2_1 = -M \sin \sigma,$ $\varphi^{y}_{x} + \varphi^{x}_{y} = 0.$	$F^2 = M^2 \cos^2 \sigma I_6$	$F\tilde{X} \overset{def}{=} \bar{X} - Mia^x \underset{x}{P'} + \{Mi\overset{y}{p}(X) - a^x\varphi^{y}_{x}\} \underset{y}{T}$

TABLE I (*continued*)

Class		Structure in V_4	Structure in $V_4 \times R^2$	F is defined by
	$D = 0$	$k^2 + K I_4 = 0$	$F^2 + K I_6 = 0.$	$F\tilde{X} \overset{def}{=} \bar{X} - \overset{x}{a}\theta^y_x \underset{y}{T}, \theta^2_1 = K, \theta^y_x + \theta^x_y = 0.$
II	$K < 0$	$k^2 + 2K I_4 = 2K \overset{x}{p} \otimes \underset{x}{P}, \underset{x}{\bar{P}} = 0$	$F^2 = 0$	$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, \overset{x}{a} \underset{x}{R} + \{\sqrt{-2K}\, \overset{y}{r}(X)$
		or		$+ a^x\theta^y_x\} \underset{y}{T}, \theta^1_1 = -\theta^2_2 = \sqrt{-2K},$
		$k^2 + 2K I_4 = 2K\overset{x}{P'} \otimes P'_x, \underset{x}{\bar{P}'} = 0$		$\theta^2_1 = \theta^1_2 = 0;$
				or
				$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, \overset{x}{a} \underset{x}{R'} + \{\sqrt{-2K}\, \overset{y}{r}'(X)$
				$+ a^x\theta^y_x\} \underset{y}{T},$
				$\theta^2_1 = -\sqrt{2K}, \theta^y_x + \theta^x_y = 0.$
			$F^2 + 2K I_6 = 0$	$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, \overset{x}{a} \underset{x}{P} + \sqrt{2K}\, \overset{x}{p}(X)\, T_x,$
				or
				$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, \overset{x}{a} \underset{x}{P'} + \sqrt{2K}\, \overset{x}{p}'(X)\, T_x.$

$K > 0$	$k^2 + 2K\, I_4 = 2K\, \overset{x}{r} \otimes \underset{x}{R},\ \underset{x}{\bar{R}} = 0$ or $k^2 + 2K\, I_4 = 2K\overset{x}{r}{}' \otimes \underset{x}{R}',\ \underset{x}{\bar{R}}' = 0$	$F^2 = 0$	$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, \overset{x}{a}\underset{x}{P} + \{\sqrt{-2K}\, \overset{y}{p}(X) - a^x\theta^y_x\}\, \underset{y}{T},\ \theta^1_1 = -\theta^2_2 = \sqrt{2K},\ \theta^2_1 = \theta^1_2 = 0,$ or $F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{-2K}\, \overset{x}{a}\underset{x}{P}' + \{\sqrt{-2K}\, \overset{y}{p}{}'(X) - a^x\theta^y_x\}\, \underset{y}{T},\ \theta^2_1 = \sqrt{2K},\ \theta^y_x + \theta^x_y = 0$
		$F^2 + 2K\, I_6 = 0$	$F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, \overset{x}{a}\underset{x}{R} + \sqrt{2K}\, \overset{x}{r}(X)\, \underset{x}{T},$ or $F\tilde{X} \overset{def}{=} \bar{X} - \sqrt{2K}\, \overset{x}{a}\underset{x}{R}' + \sqrt{2K}\, \overset{x}{r}{}'(X)\, \underset{x}{T}$

Null Electromagnetic Field

Class	Structure	Structure in $V_4 \times R$	F is defined by
III	$k^2 = \overset{1}{r} \otimes \underset{2}{R},\ \underset{2}{\bar{R}} = 0$	$F^2 = 0$	$F\tilde{X} \overset{def}{=} \bar{X} + aR_2 - \overset{1}{r}(X)\, t$
IV	$k^2 = 0$	$F^2 = 0$	$F\tilde{X} \overset{def}{=} \bar{X}$

of V_4 is given by (2.10) and $\underset{0}{N'} = 0$ when the structure of V_4 is given by (2.11). When these conditions are satisfied we have $\overset{x}{N} = 0$ and $\overset{x}{N'} = 0$ respectively.

By straight-forward calculations, we have, when F is defined by (4.2)

$$[F, F](X, \underset{x}{T}) = \underset{x0}{N}X + (\underset{x}{\overset{y}{N}}X)\underset{y}{T} \qquad \ldots(4.11a)$$

where

$$\underset{x0}{N}X \overset{def}{=} (\underset{\underset{x}{MR'}}{L}k)X - (X\theta^y_x)\underset{y}{MR'} \qquad \ldots(4.11b)$$

$$\underset{x}{\overset{y}{N}}X \overset{def}{=} -\bar{X}\theta^y_x + \underset{x}{MR'}(M\overset{y}{r'}(X)) + M\overset{y}{r'}([X, \underset{x}{MR'}]) - (X\theta^z_x)\theta^y_z$$

$$= -\bar{X}\theta^y_x - \theta^y_z X\theta^z_x + (\underset{\underset{x}{MR'}}{L}M\overset{y}{r'})(X) \qquad \ldots(4.11c)$$

$$\theta^2_1 \overset{def}{=} Mi\cos\sigma,\ \theta^y_x + \theta^x_y = 0. \qquad \ldots(4.11d)$$

$$[F, F](\underset{x}{T}, \underset{y}{T}) = [\underset{x}{MR'}, \underset{y}{MR'}] + \underset{xy}{\overset{z}{N}}\underset{z}{T} \qquad \ldots(4.12a)$$

where

$$\underset{xy}{\overset{z}{N}} \overset{def}{=} \underset{x}{MR'}\theta^z_y - \underset{y}{MR'}\theta^z_x\,\cdot \qquad \ldots(4.12b)$$

When F is defined by (4.3), we have

$$[F, F](X, \underset{x}{T}) = \underset{x0}{N'}X + (\underset{x}{\overset{y}{N'}}X)\underset{y}{T} \qquad \ldots(4.13a)$$

where

$$\underset{x0}{N'}X \overset{def}{=} i(\underset{\underset{x}{MP'}}{L}k)X - i(X\theta^y_x)\underset{y}{MP'} \qquad \ldots(4.13b)$$

$$\underset{x}{\overset{y}{N'}}X \overset{def}{=} -\bar{X}\theta^y_x - \underset{y}{MP'}(M\overset{y}{p'})(X)) - M\overset{y}{p'}([X, \underset{x}{MP'}]) - (X\theta^z_x)\theta^y_z$$

$$= -\bar{X}\theta^y_x - \theta^y_z X\theta^z_x - (\underset{\underset{x}{MP'}}{L}M\overset{y}{p'})(X) \qquad \ldots(4.13c)$$

$$\theta^2_1 = -M\sin\sigma,\ \theta^y_x + \theta^x_y = 0 \qquad \ldots(4.13d)$$

$$[F, F](\underset{x}{T}, \underset{y}{T}) = -[\underset{x}{MP'}, \underset{y}{MP'}] + \underset{xy}{\overset{z}{N}}\underset{z}{T} \qquad \ldots(4.14a)$$

where

$$\underset{xy}{\overset{z}{N'}} \overset{def}{=} iM\underset{x}{P'}\theta^z_y - iM\underset{y}{P'}\theta^z_x \cdot \qquad \text{...(4.14}b\text{)}$$

We thus see that when F is given by (4.2), we have besides $\underset{0}{N}$, $\overset{x}{N}$, the vector $\underset{x0}{NX}$ and scalar $\underset{x}{\overset{y}{N}}X$, all of which vanish when the space-time V_4 is normal.

When F is given by (4.3), we have besides $\underset{0}{N'}$, $\overset{x}{N'}$, the vector $\underset{x0}{N'X}$ and scalar $\underset{x}{\overset{y}{N'}}X$ all of which again vanish when the space-time V_4 is normal.

5. Normality of V_4 in the Second Class of $'k$

We first consider the electromagnetic tensor field of the second class when $K > 0$. From (3.9), we have

$$FX = \bar{X} + \sqrt{2K}\,\overset{x}{r}(X)\,\underset{x}{T} \qquad \text{...(5.1}a\text{)}$$

or

$$FX = \bar{X} + \sqrt{2K}\,\overset{x}{r}{}'(X)\,\underset{x}{T} \qquad \text{...(5.1}b\text{)}$$

and

$$\text{(a)} \quad \underset{x}{FT} = -\sqrt{2K}\,\underset{x}{R}, \qquad \text{or} \quad \text{(b)} \quad \underset{x}{FT} = -\sqrt{2K}\,\underset{x}{R'}. \qquad \text{...(5.2)}$$

Henceforth dashes will be dropped and $\underset{x}{R}, \overset{x}{r}, \underset{x}{P}, \overset{x}{p}$ will stand for $\underset{x}{R'}, \overset{x}{r'}, \underset{x}{P'}, \overset{x}{p'}$ also. We obtain

$$[F, F]\,(X, Y) = \underset{0}{M}(X, Y) + \overset{x}{M}(X, Y)\,\underset{x}{T} \qquad \text{...(5.3}a\text{)}$$

$$\underset{0}{M}(X, Y) \overset{def}{=} [k, k]\,(X, Y) + (d\sqrt{2K}\,\overset{x}{r})\,(X, Y)\,\sqrt{2K}\,\underset{x}{R} \qquad \text{...(5.3}b\text{)}$$

$$\overset{x}{M}(X, Y) \overset{def}{=} (d\sqrt{2K}\,\overset{x}{r})\,(\bar{X}, Y) + (d\sqrt{2K}\,\overset{x}{r})\,(X, \bar{Y})$$

$$= (\underset{\bar{X}}{L}\sqrt{2K}\,\overset{x}{r})\,(Y) - (\underset{\bar{Y}}{L}\sqrt{2K}\,\overset{x}{r})\,(X) \qquad \text{...(5.3}c\text{)}$$

$$[F, F]\,(X, \underset{x}{T}) = \underset{x0}{M}X + (\underset{x}{\overset{y}{M}}X)\,\underset{y}{T} \qquad \text{...(5.4}a\text{)}$$

where

$$\underset{x0}{M}X \overset{def}{=} (\underset{\sqrt{2K}\underset{x}{R}}{L}\;k)\,X \qquad \text{...(5.4}b\text{)}$$

$$\overset{y}{\underset{x}{M}} X \overset{def}{=} \sqrt{2K}\, \underset{x}{R}(\sqrt{2K}\, \overset{y}{r}(X)) + \sqrt{2K}\, \overset{y}{r}([X, \sqrt{2K}\, \underset{x}{R}])$$

$$= (\underset{\sqrt{2K}\underset{x}{R}}{L} \sqrt{2K}\, \overset{y}{r})(X) \quad \ldots(5.4c)$$

$$[F, F]\,(\underset{x}{T}, \underset{y}{T}) = [\sqrt{2K}\, \underset{x}{R}, \sqrt{2K}\, \underset{y}{R}]. \quad \ldots(5.5)$$

When $K < 0$, we have

$$\text{(a)} \quad FX = \bar{X} + \sqrt{2K}\, \overset{x}{p}(X)\, \underset{x}{T}, \qquad \text{(b)} \quad F\underset{x}{T} = -\sqrt{2K}\, \underset{x}{P} \quad \ldots(5.6)$$

$$[F, F]\,(X, Y) = \underset{0}{M'}(X, Y) + \overset{x}{M'}(X, Y)\, \underset{x}{T} \quad \ldots(5.7a)$$

where

$$\underset{0}{M'}(X, Y) \overset{def}{=} [k, k]\,(X, Y) + (d\, \sqrt{2K}\, \overset{x}{p})\,(X, Y)\, \sqrt{2K}\, \underset{x}{P} \quad \ldots(5.7b)$$

$$\overset{x}{M'}(X, Y) \overset{def}{=} \bar{X}(\sqrt{2K}\, \overset{x}{p}(Y)) - \bar{Y}(\sqrt{2K}\, \overset{x}{p}(X))$$

$$- \sqrt{2K}\, \overset{x}{p}([\bar{X}, Y] + [X, \bar{Y}]).$$

$$= (d\, \sqrt{2K}\, \overset{x}{p})\,(\bar{X}, Y) + (d\, \sqrt{2K}\, \overset{x}{p})\,(X, \bar{Y})$$

$$= (\underset{\bar{X}}{L}\, \sqrt{2K}\, \overset{x}{p})\,(Y) - (\underset{\bar{Y}}{L}\, \sqrt{2K}\, \overset{x}{p})\,(X) \quad \ldots(5.7c)$$

$$[F, F]\,(X, \underset{x}{T}) = \underset{x0}{M'}X + (\overset{y}{\underset{x}{M'}}X)\, \underset{y}{T} \quad \ldots(5.8a)$$

where

$$\underset{x0}{M'}X = (\underset{\sqrt{2K}\underset{x}{P}}{L}\, k)\, X, \quad \ldots(5.8b)$$

$$\overset{y}{\underset{x}{M'}}X = \sqrt{2K}\, \underset{x}{P}(\sqrt{2K}\, \overset{y}{p}(X)) + \sqrt{2K}\, \overset{y}{p}([X, \sqrt{2K}\, \underset{x}{P}])$$

$$= (\underset{\sqrt{2K}\underset{x}{P}}{L}\, \sqrt{2K}\, \overset{y}{p})\,(X). \quad \ldots(5.8c)$$

$$[F, F]\,(\underset{x}{T}, \underset{y}{T}) = [\sqrt{2K}\, \underset{x}{P}, \sqrt{2K}\, \underset{y}{P}]. \quad \ldots(5.9)$$

From the above, we have the following theorem :

Theorem 5.1 — When F is given by (3.9) and $K > 0$, the condition that the space-time V_4 of second class of $'k$ be normal is that $\underset{0}{M} = 0$. When this condition is satisfied $\underset{x0}{\overset{x}{M}}$, $\underset{x}{M}$, $\underset{x}{\overset{y}{M}}$ and $[\sqrt{2K}\underset{x}{R}, \sqrt{2K}\underset{y}{R}]$ all vanish.

When F is given by (3.10) and $K < 0$, the condition that the space-time V_4 of second class of $'k$ be normal is that $\underset{0}{M'} = 0$. When this condition is satisfied $\underset{x0}{\overset{x}{M'}}$, $\underset{x}{M'}$, $\underset{x}{\overset{y}{M'}}$ and $[\sqrt{2K}\underset{x}{P}, \sqrt{2K}\underset{y}{P}]$ all vanish.

We again consider the electromagnetic tensor field of the second class when $K > 0$. From (3.12) and (3.15) we have

$$\text{(a)}\quad FX = \bar{X} + \sqrt{-2K}\overset{y}{p}(X)\underset{y}{T}, \qquad \text{(b)}\quad F\underset{x}{T} = -\sqrt{-2K}\underset{x}{P} - \theta_x^y \underset{y}{T} \qquad ...(5.10)$$

$$\theta_1^1 = -\theta_2^2 = \sqrt{-2K},\ \theta_1^2 = \theta_2^1 = 0 \text{ for } \overset{x}{p}, \underset{x}{P}$$

$$\theta_1^2 = \sqrt{-2K},\ \theta_x^y + \theta_y^x = 0 \text{ for } \overset{x}{p'}, \underset{x}{P'}.$$

Then

$$[F, F](X, Y) = \underset{0}{P}(X, Y) + \overset{x}{P}(X, Y)\underset{x}{T} \qquad ...(5.11a)$$

where

$$\underset{0}{P}(X, Y) \overset{def}{=} [k, k](X, Y) + (d\sqrt{-2K}\overset{x}{p})(X, Y)\sqrt{-2K}\underset{x}{P} \qquad ...(5.11b)$$

$$\begin{aligned}\overset{x}{P}(X, Y) &\overset{def}{=} \bar{X}(\sqrt{-2K}\overset{x}{p}(Y)) - \bar{Y}(\sqrt{-2K}\overset{x}{p}(X)) - \sqrt{-2K}\overset{x}{p}([\bar{X}, Y] \\ &\quad + [X, \bar{Y}]) + \{X(\sqrt{-2K}\overset{y}{p}(Y) - Y(\sqrt{-2K}\overset{y}{p}(X))\}\theta_y^x \\ &= (d\sqrt{-2K}\overset{x}{p})(\bar{X}, Y) + (d\sqrt{-2K}\overset{x}{p})(X, \bar{Y}) \\ &\quad + \sqrt{-2K}\{\overset{y}{p}(Y)X\theta_y^x - \overset{y}{p}(X)Y\theta_y^x\}\end{aligned} \qquad ...(5.11c)$$

$$[F, F](X, \underset{x}{T}) = (\underset{\sqrt{-2K}\underset{x}{P}}{L}k)X + \sqrt{-2K}\overset{y}{p}([X, \sqrt{-2K}\underset{x}{P}])\underset{y}{T}\,\underset{x}{T} \qquad ...(5.12)$$

$$[F, F](\underset{x}{T}, \underset{y}{T}) = [\sqrt{-2K}\underset{x}{P}, \sqrt{-2K}\underset{y}{P}] + \sqrt{-2K}\{\underset{x}{P}\theta_y^z - \underset{y}{P}\theta_x^z\}\underset{z}{T}. \qquad ...(5.13)$$

When $K < 0$, we have from (3.13) and (3.16)

$$\text{(a)} \quad FX = \bar{X} + \sqrt{-2K}\, \overset{x}{r}(X) \underset{x}{T}, \qquad \text{(b)} \quad \underset{x}{FT} = -\sqrt{-2K}\, \underset{x}{R} + \theta_x^y \underset{y}{T} \qquad \text{...(5.14)}$$

$$\theta_1^1 = \theta_2^2 = \sqrt{2K},\ \theta_1^2 = \theta_2^1 = 0 \ \text{ for } \overset{x}{r},\ \underset{x}{R},$$

$$\theta_1^2 = -\sqrt{2K},\ \theta_x^y + \theta_y^x = 0 \ \text{ for } \overset{x}{r'},\ \underset{x}{R'},$$

We have

$$[F, F](X, Y) = \underset{0}{P'}(X, Y) + \overset{x}{P'}(X, Y) \underset{x}{T}, \qquad \text{...(5.15a)}$$

where

$$\underset{0}{P'}(X, Y) \overset{def}{=} [k, k](X, Y) + (d\sqrt{-2K}\, \overset{x}{r})(X, Y) \sqrt{-2K}\, \underset{x}{R}, \qquad \text{...(5.15b)}$$

$$\begin{aligned}\overset{x}{P'}(X, Y) \overset{def}{=} & \bar{X}(\sqrt{-2K}\, \overset{x}{r}(Y)) - \bar{Y}(\sqrt{-2K}\, \overset{x}{r}(X)) \\ & - \sqrt{-2K}\, \{\overset{x}{r}([\bar{X}, Y] + [X, \bar{Y}]) \\ & + \{X(\sqrt{-2K}\, \overset{y}{r}(Y)) - Y(\sqrt{-2K}\, \overset{y}{r}(X))\} \theta_y^x \\ = & \, d(\sqrt{-2K}\, \overset{x}{r}(\bar{X}, Y) + (d\sqrt{-2K}\, \overset{x}{r})(X, \bar{Y}) \\ & + \sqrt{-2K}\, \{\overset{y}{r}(Y)\, X\theta_y^x - \overset{y}{r}(X)\, Y\theta_y^x\}. \qquad \text{...(5.15c)}\end{aligned}$$

$$[F, F](X, \underset{x}{T}) = (\underset{\sqrt{-2K}\underset{x}{R}}{L} k) X + \sqrt{-2K}\, \overset{y}{r}([X, \sqrt{-2K}\, \underset{y}{R}) \underset{x}{T}, \qquad \text{...(5.16)}$$

$$[F, F](\underset{x}{T}, \underset{y}{T}) = [\sqrt{-2K}\, \underset{x}{R}, \sqrt{-2K}\, \underset{y}{R}] + \sqrt{-2K}\, \{\underset{y}{R}\theta_y^z - \underset{y}{R}\theta_x^z\} \underset{z}{T}. \qquad \text{...(5.17)}$$

From the above discussion, we have the following theorem :

Theorem 5.2 — When F is given by (3.12) or (3.15) and $K > 0$, the condition that the space-time V_4 of second class of $'k$ be normal is $\underset{0}{P} = 0$. When this condition is satisfied $\overset{x}{P}$, $(\underset{\sqrt{-2K}\underset{x}{P}}{L} k)$, $\overset{x}{p}([X, \sqrt{-2K}\, \underset{x}{P}])$, $[\sqrt{-2K}\, \underset{x}{P}, \sqrt{-2K}\, \underset{y}{P}]$, $\underset{x}{P}\theta_y^z - \underset{y}{P}\theta_x^z$ all vanish.

When F is given by (3.13) or (3.16) and $K < 0$, the condition that the space-time V_4 of second class of $'k$ be normal is $\underset{0}{P'} = 0$. When this condition is satisfied $\overset{x}{P'}$, $(\underset{\sqrt{-2K}\underset{x}{R}}{L} k)$, $\overset{x}{r}([X, \sqrt{-2K}\underset{x}{R}])[\sqrt{-2K}\underset{x}{R}, \sqrt{-2K}\underset{y}{R}]$, $\underset{x}{R}\theta^z_y - \underset{y}{R}\theta^z_x$ all vanish.

6. Normality of V_4 in the Third Class of $'k$

We now consider the electromagnetic tensor field of the third class of $'k$. From (3.18), we have

$$\text{(a)} \quad FX = \bar{X} - \overset{1}{r}(X)\, t, \qquad \text{(b)} \quad Ft = R_2. \qquad \ldots(6.1)$$

Hence we obtain

$$[F, F]\,(X, Y) = \underset{0}{Q}(X, Y) + Q(X, Y)\, t, \qquad \ldots(6.2a)$$

where

$$\underset{0}{Q}(X, Y) \overset{def}{=} [k, k]\,(X, Y) + (d\overset{1}{r})\,(X, Y)\,\underset{2}{R}, \qquad \ldots(6.2b)$$

$$Q(X, Y) \overset{def}{=} -\, d\overset{1}{r}(\bar{X}, Y) - d\overset{1}{r}(X, \bar{Y}) = -\,(\underset{\bar{X}}{L}\overset{1}{r})\,(Y) + (\underset{\bar{Y}}{L}\overset{1}{r})\,(X). \quad \ldots(6.2c)$$

and

$$[F, F]\,(X, t) = [\bar{X}, \underset{2}{R}] - \overline{[X, \underset{2}{R}]} + \{\underset{2}{R}(\overset{1}{r}(X)) - \overset{1}{r}([X, \underset{2}{R}])\}\, t$$

$$= -\,(\underset{\underset{2}{R}}{L}k)\, X + (d\overset{1}{r})\,(\underset{2}{R}, X)\, t. \qquad \ldots(6.3)$$

From the above, we have the following theorem.

Theorem 6.1 — When F is given by (3.18), the condition that the spape-time V_4 of the third class is normal is $\underset{0}{Q} = 0$. When this condition is satisfied Q, $\underset{\underset{2}{R}}{L}k$ and $(d\overset{1}{r})\,(\underset{2}{R}, X)$ all vanish.

7. Conclusion

In sections 4, 5 and 6, we have obtained the conditions that the space-time V_4 be normal, when the electromagnetic tensor field $'k$ is non-null (is of the first and second class) and is null (of the third class). These conditions are obtained in elegant forms and have great mathematical significance. The physical interpretation of these conditions is an open question and will be given in subsequent work.

References

Eliopolous, H. (1965). On the general theory of differentiable manifolds with almost tangent structure. *Camb. Math. Bull.*, **8**, 721–28.

Hlavatý, V. (1957). Geometry of Einstein's Unified Field Theory. P. Noordhoff Ltd., Groningen.

Legrand, G. (1966). Sur les Variétés a structure de presque product complexe. *C. R. Acad. Sci., Paris*, **242**, 335–36.

Mishra, R. S. (1976). Structures in electromagnetic tensor fields. *Tensor, N. S.*, **30**, 145–156.