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SOME PERFECT FLUID COSMOLOGICAL MODELS OF PLANE SYMMETRY WITH INCIDENT MAGNETIC FIELD

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In this paper the authors have derived cosmological models of plane symmetry by using some supplementary condition between the metric potentials. Various physical properties of the models have been explored.

INTRODUCTION

Recently Roy and Prakash (1978) have derived an anisotropic magneto-hydrodynamic cosmological model in which the free gravitational field is of Petrov type D. In this paper we have obtained cosmological models of plane symmetry by using a supplementary condition between the metric potentials. The first two models reduce to that of perfect fluid distribution when the magnetic field vanishes but in the third model no such reduction is possible.

We consider the metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \qquad \ldots(1)$$

where the metric potentials A, B and C are functions of time alone. The energy momentum tensor is

$$T_i^j = (\epsilon + p)\, V_iV^j + p\delta_i^j + E_i^j \qquad \ldots(2)$$

where

$$E_i^j = \bar{\mu}\,[h_lh^l(V_iV^j + \tfrac{1}{2}\delta_i^j) - h_ih^j] \qquad \ldots(3)$$

$\bar{\mu}$ being the magnetic permeability, ϵ the density, p the pressure and

$$h_i = \frac{1}{2\bar{\mu}}\, V^j \sqrt{-g}\, \epsilon_{jikm}F^{km} \quad \text{(Lichnerowicz 1967)}$$

F_{ij} being the electromagnetic field tensor and V^i the flow vector satisfying

$$g_{ij}V^iV^j = -1. \qquad \ldots(4)$$

The coordinates are assumed to be comoving so that

$$V^1 = V^2 = V^3 = 0 \text{ and } V^4 = \frac{1}{A}.$$

We assume that F_{23} is the only non-vanishing component of the electromagnetic field tensor F_{ij}. The first set of Maxwell's equation

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

leads to F_{23} being a constant say H.

The field equations

$$-8\pi T_i^j = R_i^j - \tfrac{1}{2} R\delta_i^j + \Lambda \delta_i^j$$

for the line element (1) are

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC}\right] - \Lambda = 8\pi\left[p - \frac{H^2}{2\bar{\mu}B^2C^2}\right] \qquad ...(5)$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi\left[p + \frac{H^2}{2\bar{\mu}B^2C^2}\right] \qquad ...(6)$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi\left[p + \frac{H^2}{2\bar{\mu}B^2C^2}\right] \qquad ...(7)$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{A_4C_4}{AC}\right] + \Lambda = 8\pi\left[\epsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right]. \qquad ...(8)$$

The suffix 4 after the symbols A, B and C indicates ordinary differentiation with respect to time.

Solution of the Field Equations

Equations (5) – (8) are four equations in five unknowns A, B, C, ϵ and p. For complete solution we need an extra condition. We assume that

$$A = B^nC^n \qquad ...(9)$$

where n is a positive constant. Clearly $n \neq \frac{1}{2}$ as otherwise H will be zero. From eqns. (5), (6) and (9) we have

$$\frac{\mu_{44}}{\mu} + \left(\frac{\nu_4}{\nu}\right)_4 + \frac{\mu_4}{\mu}\left(\frac{\nu_4}{\nu}\right) = -\frac{16\pi H^2\mu^{2n-2}}{(2n-1)\bar{\mu}} \qquad ...(10)$$

and from eqns. (6) and (7) we get

$$\frac{\nu_4}{\nu} = \frac{h}{\mu} \qquad ...(11)$$

h being a constant of integration and $BC = \mu$, $\frac{B}{C} = \nu$. From eqns. (10) and (11) we have

$$\frac{\mu_{44}}{\mu} = -\frac{m^2}{(2n-1)}\mu^{2n-2} \quad ...(12)$$

where

$$m^2 = \frac{16\pi H^2}{\bar{\mu}}.$$

Case I — Let $n > \frac{1}{2}$. Equation (12) on integration gives

$$\frac{d\mu}{dt} = \sqrt{b^2 - \frac{m^2\mu^{2n}}{n(2n-1)}} \quad ...(13)$$

b^2 being a constant of integration. From eqns. (11) and (13) we have

$$\nu = \alpha\left\{\frac{1}{4b^2n(2n-1)}\right\}^{-h/2bn}\left[m^2\left\{\frac{b+f(\mu)}{b-f(\mu)}\right\}\right]^{-h/2bn} \quad ...(14)$$

where α is an arbitrary constant and $f^2(\mu) = b^2 - \frac{m^2\mu^{2n}}{n(2n-1)}$. By suitable transformation of co-ordinates metric (1) takes the form

$$ds^2 = T^{2n}\left[dx^2 - \frac{dT^2}{F}\right] + T\left[m^2\left\{\frac{b+\sqrt{F}}{b-\sqrt{F}}\right\}\right]^{-\beta} dy^2$$
$$+ T\left[m^2\left\{\frac{b+\sqrt{F}}{b-\sqrt{F}}\right\}\right]^{\beta} dz^2 \quad ...(15)$$

where $\beta = \frac{h}{2bn}$ and $F = b^2 - \frac{m^2T^{2n}}{n(2n-1)}$.

Case II — Let $0 < n < \frac{1}{2}$. Equation (12) on integration gives

$$\frac{d\mu}{dt} = \sqrt{a + \frac{m^2\mu^{2n}}{n(1-2n)}} \quad ...(16)$$

a being constant of integration. Two cases arise.

Sub Case (IIa) — Let $a > 0$ say $a = b^2$. The metric in this case is given by

$$ds^2 = T^{2n}\left[dx^2 - \frac{dT^2}{F'}\right] + T\left[m^2\left\{\frac{\sqrt{F'}+b}{\sqrt{F'}-b}\right\}\right]^{-\beta} dy^2$$
$$+ T\left[m^2\left\{\frac{\sqrt{F'}+b}{\sqrt{F'}-b}\right\}\right]^{\beta} dz^2 \quad ...(17)$$

where $F' = b^2 + \frac{m^2T^{2n}}{n(1-2n)}$.

Sub Case (IIb) — Let $a < 0$ say $a = -d^2$. From eqns. (11) and (16) we have

$$\nu = M\exp\left[\frac{h}{nd}\tan^{-1}\left\{\frac{m^2\mu^{2n}}{n(1-2n)\,d^2} - 1\right\}^{1/2}\right] \quad ...(18)$$

M being a constant of integration. By suitable transformation of co-ordinates, the line element (1) takes the form

$$ds^2 = T^{2n}\left[dx^2 - \frac{dT^2}{\dfrac{m^2T^{2n}}{n(1-2n)} - d^2}\right]$$
$$+ T\exp\left[q\tan^{-1}\left\{\frac{m^2T^{2n}}{n(1-2n)d^2} - 1\right\}^{1/2}\right]dy^2$$
$$+ T\exp\left[-q\tan^{-1}\left\{\frac{m^2T^{2n}}{n(1-2n)\,d^2} - 1\right\}^{1/2}\right]dz^2 \qquad ...(19)$$

where $q = \dfrac{h}{nd}$.

Some Physical and Geometrical Features

The pressure and density for the model (15) are given by

$$8\pi p = \frac{b^2}{4}\left[(4n+1) - 4n^2\beta^2\right]T^{-2n-2} + \frac{n-1}{4n}\cdot\frac{2n+1}{2n-1}m^2T^{-2} - \Lambda \qquad ...(20)$$

$$8\pi\epsilon = \frac{b^2}{4}\left[(4n+1) - 4n^2\beta^2\right]T^{-2n-2} - \frac{n+1}{4n}\cdot\frac{2n+1}{2n-1}m^2T^{-2} + \Lambda. \qquad ...(21)$$

The scalar of expansion $\theta = V^i_{;i}$ for the flow vector V^i is given by

$$\theta = \frac{n+1}{T^{n+1}}\sqrt{b^2 - \frac{m^2T^{2n}}{n(2n-1)}}. \qquad ...(22)$$

The rotation ω_{ij} is identically zero and the shear is given by

$$\sigma^2 = \tfrac{1}{6}\left[3h^2 + (2n-1)^2\left\{b^2 - \frac{m^2T^{2n}}{n(2n-1)}\right\}\right]T^{-2n-2}. \qquad ...(23)$$

The red shift in the model (15) is given by

$$\frac{\lambda + d\lambda}{\lambda} = \frac{T^n\{b^2 - \psi T_2^{2n}\}^{1/2}\left[\dfrac{T_1^{(2n-1)/2}\left\{m^2\left(\dfrac{b+\sqrt{b^2-\psi T_1^{2n}}}{b-\sqrt{b^2-\psi T_1^{2n}}}\right)\right\}^{-\beta/2}}{\{b^2-\psi T_1^{2n}\}^{1/2}} + U_z\right]}{T_2^{(2n-1)/2}\left[m^2\left\{\dfrac{b+\sqrt{b^2-\psi T_2^{2n}}}{b-\sqrt{b^2-\psi T_2^{2n}}}\right\}\right]^{-\beta/2}[T^{2n} - U^2\{b^2 - \psi T^{2n}\}]^{1/2}} \qquad ...(24)$$

where $\psi = \dfrac{m^2}{n(2n-1)}$, U is the velocity of source at the time of emission and U_z is the z component of the velocity. The non-vanishing components of the conformal curvature tensor are given by

$$C_{12}^{12} = -\frac{1}{12}\left[\left\{(2n-1)\,b^2 + h^2 + 6nh\left(b^2 - \frac{m^2T^{2n}}{n(2n-1)}\right)^{1/2}\right\} T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right] \quad \text{...(25)}$$

$$C_{13}^{13} = -\frac{1}{12}\left[\left\{(2n-1)\,b^2 + h^2 - 6nh\left(b^2 - \frac{m^2T^{2n}}{n(2n-1)}\right)^{1/2}\right\} T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right] \quad \text{...(26)}$$

$$C_{23}^{23} = \tfrac{1}{6}\left[\{(2n-1)\,b^2 + h^2\}\,T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right]. \quad \text{...(27)}$$

Hence the space time is non-degenerate Petrov type I when $h \neq 0$. When $h = 0$, it is type D. The universe has singularity at $T = 0$. Thus the universe expands from the singular state and goes up to the co-ordinate time

$$T = \left\{\frac{b^2n(2n-1)}{m^2}\right\}^{1/2n}.$$

The latter value is not the singularity of the universe because at that point the density, pressure and the invariants of curvature tensor are finite. In the absence of the magnetic field the expansion continues up to $T = \infty$ at which time the universe becomes an infinite flat space time. The shear remains positive throughout, but in the absence of the magnetic field it tends to zero as $T \to \infty$. If l be the linear dimension of the universe we find from the relation

$$\frac{\dot{l}}{l} = \tfrac{1}{3}\theta \quad \text{...(28)}$$

where

$$\dot{l} = l_{,\alpha}V^{\alpha}$$

that

$$l = \xi T^{(n+1)/3} \quad \text{...(29)}$$

ξ being a positive constant. Thus the linear dimension in the universe is independent of the magnetic field at any instant T. $\dot{l} = 0$ when $T = \left\{\frac{b^2n(2n-1)}{m^2}\right\}^{1/2n}$. The expansion stops at that instant. The deceleration parameter q defined by

$$q = -\frac{(l_{,\alpha}V^{\alpha})_{,\beta}\,V^{\beta}\,.\,l}{(l_{,\alpha}\,V^{\alpha})^2} \quad \text{...(30)}$$

has the value

$$q = \frac{3}{n+1}\left[\frac{m^2T^{2n}}{(2n-1)\left\{b^2 - \frac{m^2T^{2n}}{n(2n-1)}\right\}} + \tfrac{2}{3}(n+1)\right]. \quad \text{...(31)}$$

since $q > 0$, the rate of expansion is a decreasing function of time. The magnetic field contributes negative terms to the expansion and shear and positive term to the deceleration. Hence its effect is to decrease the expansion as well as the shear. In order to be physically meaningful the model has to satisfy reality conditions (Ellis 1971)

$$\text{(i)}\quad \epsilon + p > 0, \qquad \text{(ii)}\quad \epsilon + 3p > 0, \qquad \text{(iii)}\quad \epsilon - p \geqslant 0.$$

The condition (iii) assures that the velocity of sound cannot exceed the velocity of light. Condition (i) requires that

$$\beta^2 < \frac{4n+1}{4n^2}. \qquad \text{...(32)}$$

When this is satisfied, we must have

$$T^{2n} < \frac{n(2n-1)\, b^2\, [4n + 1 - 4n^2\beta^2]}{m^2(2n+1)}. \qquad \text{...(33)}$$

It is also clear from the metric (15) that

$$T^{2n} < \frac{b^2 n(2n-1)}{m^2}. \qquad \text{...(34)}$$

Thus T^{2n} is less than the smaller among the expressions on the right hand side of (33) and (34). Condition (ii) leads to

$$b^2\, [4n + 1 - 4n^2\beta^2]\, T^{-2n-2} + \frac{(n-2)\,(2n+1)}{2n(2n-1)}\, m^2 T^{-2} - 2\Lambda > 0 \qquad \text{...(35)}$$

which is a condition involving Λ.

Condition (iii) cannot be satisfied when $\Lambda \leqslant 0$. When $\Lambda > 0$ it gives

$$T^2 \geqslant \frac{m^2(2n+1)}{4\,\Lambda\,(2n-1)}. \qquad \text{...(36)}$$

When the magnetic field tends to zero, the metric (15) takes the form

$$ds^2 = T^{2n}\left[dx^2 - \frac{dT^2}{b^2}\right] + T^{1+2n\beta}\, dy^2 + T^{1-2n\beta}\, dz^2 \qquad \text{...(37)}$$

after suitable transformation of co-ordinates. This metric represents a perfect fluid distribution for which

$$8\pi p = \frac{b^2}{4}\, [(4n+1) - 4n^2\beta^2]\, T^{-2n-2} - \Lambda \qquad \text{...(38)}$$

$$8\pi\epsilon = \frac{b^2}{4}\, [(4n+1) - 4n^2\beta^2]\, T^{-2n-2} + \Lambda. \qquad \text{...(39)}$$

The reality conditions (i), (ii) and (iii) are satisfied when (32) holds and $\Lambda \geqslant 0$ and in this case we have

$$T^{2n+2} < \frac{b^2 [4n + 1 - 4n^2\beta^2]}{2\Lambda}. \qquad ...(40)$$

The pressure, density and the kinematical parameters for the metric (17) are same as those of model (15). This model starts expanding from its singular state at $T = 0$ and continues to expand till $T = \infty$. The reality conditions however restrict the time period during which the model exists. Condition (i) is identically satisfied when $4n + 1 - 4n^2\beta^2 < 0$. If this be negative, then

$$T^{2n} > -\frac{n(1 - 2n)\, b^2 [4n + 1 - 4n^2\beta^2]}{m^2(2n + 1)}. \qquad ...(41)$$

Condition (ii) gives the same expression as in (35) which is a condition involving Λ. Condition (iii) is identically satisfied when $\Lambda \geqslant 0$. If $\Lambda < 0$ then we have

$$T^2 \leqslant -\frac{m^2(2n + 1)}{4\Lambda(1 - 2n)}. \qquad ...(42)$$

The pressure and density for the model (19) are given by

$$8\pi p = -\frac{d^2}{4}[4n + 1 + n^2q^2]\, T^{-2n-2} - \frac{n - 1}{4n} \cdot \frac{2n + 1}{1 - 2n} m^2 T^{-2} - \Lambda \qquad ...(43)$$

$$8\pi\epsilon = -\frac{d^2}{4}[4n + 1 + n^2q^2]\, T^{-2n-2} + \frac{n + 1}{4n}\, \frac{2n + 1}{1 - 2n} m^2 T^{-2} + \Lambda. \qquad ...(44)$$

The scalar of expansion $\theta = V^i_{;i}$ for the flow vector V^i is given by

$$\theta = \frac{n + 1}{T^{n+1}} \sqrt{\frac{m^2 T^{2n}}{n(1 - 2n)} - d^2}. \qquad ...(45)$$

The rotation ω_{ij} is identically zero and the shear is given by

$$\sigma^2 = \tfrac{1}{6}\left[3h^2 + (1 - 2n)^2 \left\{\frac{m^2 T^{2n}}{n(1 - 2n)} - d^2\right\}\right] T^{-2n-2}. \qquad ...(46)$$

The red shift in the model (19) is given by

$$\frac{\lambda + d\lambda}{\lambda} = \frac{T^n(\phi T_2^{2n} - d^2)^{1/2}\left[T_1^{(2n-1)/2} \exp\left(\tfrac{1}{2} q \tan^{-1}\{\phi T_1^{2n} - d^2\}^{1/2}\right) . (\phi T_1^{2n} - d^2)^{-1/2} + U_z\right]}{T_2^{(2n-1)/2} \exp\left(\tfrac{1}{2} q \tan^{-1}\{\phi T_2^{2n} - d^2\}^{1/2}\right) (T^{2n} - U^2\{\phi T^{2n} - d^2\})^{1/2}} \qquad ...(47)$$

where $\phi = \dfrac{m^2}{n(1-2n)}$. The non-vanishing components of conformal curvature tensor are given by

$$C^{12}_{12} = -\frac{1}{12}\left[\left\{(1-2n)\,d^2 + h^2 + 6nh\left(\frac{m^2T^{2n}}{n(1-2n)} - d^2\right)^{1/2}\right\} T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right] \qquad ...(48)$$

$$C^{13}_{13} = -\frac{1}{12}\left[\left\{(1-2n)\,d^2 + h^2 - 6nh\left(\frac{m^2T^{2n}}{n(1-2n)} - d^2\right)^{1/2}\right\} T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right] \qquad ...(49)$$

$$C^{23}_{23} = \tfrac{1}{6}\left[\{(1-2n)\,d^2 + h^2\}\,T^{-2n-2} + \frac{n-1}{n}\,m^2T^{-2}\right]. \qquad ...(50)$$

Hence the space time is non-degenerate Petrov type I when $h \neq 0$. When $h = 0$ it is of Petrov type D. There is an apparent singularity at $T = \left\{\dfrac{d^2n(1-2n)}{m^2}\right\}^{1/2n}$. But it is not a real singularity because the density, pressure and the invariants of the curvature tensor are finite. At this instant the expansion is zero but its rate is positive and it continues to be positive for $T < \left\{\dfrac{d^2(n+1)\,(1-2n)}{m^2}\right\}^{1/2n}$. After this stage it becomes negative and finally the expansion stops at $T = \infty$, at which stage the geometry of the model is flat. Thus the universe starts expanding from its non-singular stage at time $T = \left\{\dfrac{d^2n(1-2n)}{m^2}\right\}^{1/2n}$ till it becomes an infinite flat space when it stops expanding, the model remaining non-singular throughout the period of its expansion. The linear dimension l for this model is given by

$$l = \gamma T^{(n+1)/3} \qquad ...(51)$$

where γ is a positive constant. The linear dimension is independent of the magnetic field as in (15). The deceleration parameter q is given by

$$q = -\frac{3}{n+1}\left[\frac{m^2T^{2n}}{\left\{\dfrac{m^2T^{2n}}{n(1-2n)} - d^2\right\}} - \tfrac{2}{3}(n+1)\right]. \qquad ...(52)$$

The reality condition (i) in this case requires that

$$T^{2n} > \frac{n(1-2n)\,d^2\,[4n+1+n^2q^2]}{m^2(2n+1)}. \qquad ...(53)$$

Also it is clear from the metric (19) that

$$T^{2n} > \frac{d^2n(1-2n)}{m^2}. \qquad ...(54)$$

Hence T^{2n} is greater than the larger among the right-hand sides of (53) and (54). Condition (ii) leads to

$$- d^2 [4n + 1 + n^2q^2] T^{-2n-2} - \frac{n-2}{2n} \cdot \frac{2n+1}{1-2n} m^2 T^{-2} - 2\Lambda > 0 \qquad ...(55)$$

which is a condition involving Λ. Condition (iii) is identically satisfied when $\Lambda \geqslant 0$. If $\Lambda < 0$, then

$$T^2 \leqslant - \frac{m^2(2n+1)}{4\Lambda(1-2n)}. \qquad ...(56)$$

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