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HYPERSURFACES OF ALMOST HYPERBOLIC HERMITE MANIFOLDS

R. B. PAL AND R. S. MISHRA

Department of Mathematics, Banaras Hindu University, Varanasi 221005

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In this paper it is shown that a hypersurface of hyperbolic Hermite manifold admits an induced hyperbolic contact structure. A condition for hypersurface of an almost hyperbolic Hermite manifold with vanishing curvature tensor to be conformally flat is also obtained.

1. INTRODUCTION

Let V_m be an m-dimensional differentiable manifold in which there exist a vector-valued linear function F and a Riemannian metric G, satisfying

$$F^2\lambda = \lambda \qquad \text{...(1.1)}$$

$$G(F\lambda, F\mu) = -\, G(\lambda, \mu) \qquad \text{...(1.2)}$$

for arbitrary vector fields λ, μ in V_m. Then V_m is called an almost hyperbolic Hermite manifold (Prvanović 1971, Dube 1973).

An almost hyperbolic Hermite manifold V_m for which

$$(E_\lambda\, F)\, \mu = 0 \qquad \text{...(1.3)}$$

where E is the Riemannian connexion, is satisfied, is called hyperbolic Kählerian manifold (Prvanović 1971).

Let V_n be a differentiable manifold of dimension n. Let there exist a tensor field f of the type (1, 1), a 1-form A, a vector field T and the Riemannian metric tensor g satisfying

$$f^2X = X + A(X)\, T \qquad \text{...(1.4)}$$

$$fT = 0 \qquad \text{...(1.5)}$$

$$A(fX) = 0 \qquad \text{...(1.6)}$$

$$A(T) = -1 \qquad \text{...(1.7)}$$

$$g(X, T) = A(X) \qquad \text{...(1.8)}$$

$$g(fX, fY) = -g(X, Y) - A(X)\, A(Y). \qquad \text{...(1.9)}$$

Then V_n is called hyperbolic contact metric manifold (Upadhyay and Dube 1973).

The Nijenhuis tensor $N(X, Y)$ of the hyperbolic contact metric manifold is given by the relation

$$N(X, Y) = (D_{fX}f)\, Y - (D_{fY}f)\, X + f(D_Y f)\, X - f(D_X f)\, Y. \quad ...(1.10)$$

D is the Riemannian connexion of V_n.

Definition 1.1 — When the tensor field

$$P(X, Y) = N(X, Y) + (dA)\,(X, Y)\, T \quad ...(1.11)$$

vanishes, where N is the Nijenhuis tensor formed with f then hyperbolic contact metric manifold V_n is said to be normal.

2. Hypersurfaces

Let us consider a hypersurface V_n, $(n = m - 1)$ of an almost hyperbolic Hermite manifold with the immersion map $b : V_n \to V_m$ such that a point

$$p \in V_n \Rightarrow bp \in V_m.$$

Let B be the corresponding Jacobian map, such that a vector field X in V_n at $p \Rightarrow BX$ in V_m at bp. Let g be the induced Riemannian metric of V_n. Let N be a unit normal vector to V_n. Then we have

$$\left.\begin{array}{ll} \text{(a)} & G(BX, BY)\, ob = g(X, Y) \\ \text{(b)} & G(BX, N)\, ob = 0 \\ \text{(c)} & G(N, N) = 1. \end{array}\right\} \quad ...(2.1)$$

Let us express the transformation of BX and N by F as the sum of tangential and normal parts in the form

$$\left.\begin{array}{ll} \text{(a)} & FBX = BfX + A(X)\, N \\ \text{(b)} & FN = -BT. \end{array}\right\} \quad ...(2.2)$$

Theorem 2.1 — A hypersurface V_n of an almost hyperbolic Hermite manifold V_m is a hyperbolic contact metric manifold.

PROOF : Premultiplying (2.2*a*) and (2.2*b*) by F and using (1.1) and (2.2) and collecting tangential and normal parts, we have

$$\text{(a)} \quad f^2X = X + A(X)\, T, \qquad \text{(b)} \quad A(fX) = 0. \quad ...(2.3)$$

$$\text{(a)} \quad fT = 0, \qquad \text{(b)} \quad A(T) = -1. \quad ...(2.4)$$

Also from (1.2), we have

$$G(FBX, FBY) = -G(BX, BY). \quad ...(2.5)$$

Making use of (2.1) and (2.2) in (2.5), we have

$$g(fX, fY) = -g(X, Y) - A(X)\,A(Y). \qquad ...(2.6)$$

Equations (2.3), (2.4) and (2.6) prove the statement.

Equations of Gauss and Weingarten are given by

$$\left.\begin{array}{ll}\text{(a)} & E_{BX}BY = BD_XY + {}'H(X, Y)\,N \\ \text{(b)} & E_{BX}N = -\,BHX\end{array}\right\} \qquad ...(2.7)$$

where E and D are Riemannian connexions in V_m and V_n respectively. $'H(X, Y)$ is a symmetric second fundamental tensor with respect to normal N and $H(X)$ is a tensor field of the type (1, 1) defined by

$$'H(X, Y) \stackrel{def}{=} g(HX, Y). \qquad ...(2.8)$$

Let us suppose that V_m be a hyperbolic Kählerian manifold, then (1.3) implies.

$$(E_{BX}F)\,BY = 0 \Rightarrow E_{BX}FBY = FE_{BX}BY. \qquad ...(2.9)$$

Substituting in (2.9) from (2.2) and (2.7) and collecting tangential and normal parts, we get

$$\left.\begin{array}{ll}\text{(a)} & (D_Xf)\,Y = -\,'H(X, Y)\,T + A(Y)\,HX \\ \text{(b)} & (D_XA)\,(Y) = -\,'H(X, fY).\end{array}\right\} \qquad ...(2.10)$$

Theorem 2.2 — If V_n is a hypersurface of a hyperbolic Kählerian manifold V_m and H commutes with f, then hyperbolic contact metric manifold V_n is normal.

PROOF : Rewrite (1.11)

$$P(X, Y) = N(X, Y) + (dA)\,(X, Y)\,T.$$

In consequence of (1.10), we have

$$P(X, Y) = (D_{fX}f)\,Y - (D_{fY}f)\,X + f(D_Yf)\,X - f(D_Xf)\,Y$$
$$+ \{(D_XA)\,(Y) - (D_YA)\,(X)\}\,T. \qquad ...(2.11)$$

In consequence of (2.10), we have

$$P(X, Y) = A(Y)\,\{HfX - fHX\} - A(X)\,\{HfY - fHY\}. \qquad ...(2.12)$$

If H commutes with f, then we have $P(X, Y) = 0$.

Hence, we have proved the theorem.

Let R be the curvature tensor of the Riemannian connexion E and K be the curvature tensor of the induced Riemannian connexion D, then we have

$$'R(BX, BY, BZ, BU) = 'K(X, Y, Z, U) + 'H(X, Z)\,'H(Y, U) - 'H(Y, Z)\,'H(X, U) \quad ...(2.13a)$$

$$'R(BX, BY, BZ, N) = (D_X'H)(Y, Z) - (D_Y'H)(X, Z) \quad ...(2.13b)$$

where

$$'R(BX, BY, BZ, BU) \overset{def}{=} G(R(BX, BY, BZ), BU) \quad ...(2.14a)$$

$$'K(X, Y, Z, U) \overset{def}{=} g(K(X, Y, Z), U). \quad ...(2.14b)$$

Theorem 2.3 — A hypersuface V_n of an almost hyperbolic Hermite manifold V_m with vanishing curvature tensor be conformally flat if $'H(X, Y) = -g(fX, fY)$.

PROOF : Since the curvature tensor of an almost hyperbolic Hermite manifold V_m vanishes, then from (2.13*a*), we have

$$'K(X, Y, Z, U) = 'H(X, U)\,'H(Y, Z) - 'H(X, Z)\,'H(Y, U) \quad ...(2.15)$$

from the supposition and using (1.9), we have

$$'K(X, Y, Z, U) = g(Y, Z)\,g(X, U) - g(X, Z)\,g(Y, U) + g(Y, Z)\,A(X)\,A(U) - g(X, Z)\,A(Y)\,A(U) + g(X, U)\,A(Y)\,A(Z) - g(Y, U)\,A(X)\,A(Z) \quad ...(2.16a)$$

or

$$K(X, Y, Z) = g(Y, Z)\,X - g(X, Z)\,Y + g(Y, Z)\,A(X)\,T - g(X, Z)\,A(Y)\,T + A(Y)\,A(Z)\,X - A(X)\,A(Z)\,Y. \quad ...(2.16b)$$

Contracting the above equation with respect to X, we set

$$\text{Ric}\,(Y, Z) = (n - 2)\,(g(Y, Z) + A(Y)\,A(Z)) \quad ...(2.17a)$$

or

$$R(Y) = (n - 2)\,(Y + A(Y)\,T). \quad ...(2.17b)$$

Contracting the above equation with respect to Y, we get

$$R = (n - 2)\,(n - 1) \quad ...(2.18)$$

where R is a scalar curvature tensor of V_n.

On making use of (2.17) and (2.18) in (2.16), we get

$$K(X, Y, Z) - \frac{1}{(n - 2)}\{\text{Ric}\,(Y, Z)\,X - \text{Ric}\,(X, Z)\,Y + g(Y, Z)\,R(X) -$$

(*equation continued on p.* 632)

$$- g(X, Z)\, R(Y)\} - \frac{R}{(n-1)(n-2)} \{g(Y, Z)\, X - g(X, Z)\, Y\} = 0$$

$$\Rightarrow V(X, Y, Z) = 0$$

where $V(X, Y, Z)$ is a conformal curvature tensor. Hence, we have the statement.

Theorem 2.4 — In a hyperbolic contact metric hypersurface V_n of a hyperbolic Kählerian monifold V_m, the induced structure tensor f is covariant constant if HX vanishes.

PROOF : Rewrite (2.10*a*)

$$(D_X f)\, Y = -\, 'H(X, Y)\, T + A(Y)\, HX.$$

If we suppose HX vanishes, then we have

$$(D_X f)\, Y = 0$$

Hence, we have the statement.

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