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ON A QUASI-NORMAL ($\psi, g, u, v, e, \lambda$)-STRUCTURE

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In the present paper it is shown that the ($\psi, g, u, v, e, \lambda$)-structure is quasi-normal in the sense of Yano (Yano and Ki 1972) iff the vector fields are harmonic. Also, the induced structure of the invariant submanifold of co-dimension 2 is quasi-normal.

1. GENERAL FORMULA FOR ($\psi, g, u, v, e, \lambda$)-STRUCTURE

Let V_n be a C^∞ differentiable manifold with (1, 1) tensor field ψ, Riemannian metric tensor g, three 1-forms u, v, e and associated with three vector fields U, V, E and a differentiable function λ satisfying (Baik 1972, Yano 1965)

(a) $\psi_j^t \psi_t^h = -\delta_j^h + u_j u^h + v_j v^h + e_j e^h,$

(b) $\psi_j^t \psi_i^s g_{ts} = g_{ji} - u_j u_i - v_j v_i - e_j e_i,$

(c) $\psi_i^t u_t = \lambda v_i, \ \psi_i^t v_t = -\lambda u_i, \ \psi_i^t e_t = 0,$

(d) $\psi_i^h u^i = -\lambda v^h, \ \psi_i^h v^i = \lambda u^h, \ \psi_i^h e^i = 0,$

(e) $u_i u^i = 1 - \lambda^2, \ u_i v^i = 0, \ u_i e^i = 0,$

(f) $v_i u^i = 0, \ v_i v^i = 1 - \lambda^2, \ v_i e^i = 0,$

(g) $e_i u^i = 0, \ e_i v^i = 0, \ e_i e^i = 1.$ ...(1.1)

Let us put

$$\psi_{ji} = \psi_j^t g_{ti} \quad \text{(being skew symmetric)} \qquad ...(1.2)$$

$h, i, j, \ldots$ run over $1, 2, \ldots n = (2m + 3)$.

Let us define (1.2) type of torsion tensor as follows:

$$S_{ji}^h \stackrel{def}{=} \psi_j^t \nabla_t \psi_i^h - \psi_i^t \nabla_t \psi_j^h - \psi_t^h (\nabla_j \psi_i^t - \nabla_i \psi_j^t) + u_{ji} u^h + v_{ji} v^h + e_{ji} e^h \qquad ...(1.3)$$

where $u_{ji} = \nabla_j u_i - \nabla_i u_j$ is covariant differential w.r.t. Levi Civita connection. If the torsion tensor S_{ji}^h vanishes, the ($\psi, g, u, v, e, \lambda$)-structure is called normal.

Transvecting (1.3) with u_h, v_h, e_h and also keeping in view (1.1) we obtain

$$S_{ji}^{h}\, u_h = u_{ji} + \lambda(u_i \nabla_j \lambda - u_j \nabla_i \lambda) + \lambda(\psi_j^{t}\, v_{ti} - \psi_i^{t}\, v_{tj})$$

$$+ \nabla_t \lambda(\psi_j^{t}\, v_i - \psi_i^{t}\, v_j) - \psi_i^{s}\, \psi_j^{t}\, u_{ts}, \qquad \ldots(1.4)$$

$$S_{ji}^{h}\, v_h = v_{ji} + \lambda(v_i \nabla_j \lambda - v_j \nabla_i \lambda) - \lambda(\psi_j^{t}\, u_{ti} - \psi_i^{t}\, u_{tj})$$

$$- \nabla_t \lambda(\psi_j^{t}\, u_i - \psi_i^{t}\, u_j) - \psi_j^{t}\, \psi_i^{s}\, v_{ts}, \qquad \ldots(1.5)$$

and

$$S_{ji}^{h}\, e_h = e_{ji} + \psi_j^{t}\, \psi_i^{s}\, e_{st}. \qquad \ldots(1.6)$$

We have (Yano and Ki 1972)

$$\psi_{jih} = \nabla_j \psi_{ih} + \nabla_i \psi_{hj} + \nabla_h \psi_{ji}. \qquad \ldots(1.7)$$

Using (1.3) and (1.7) the covariant components S_{jih} can be written as

$$S_{jih} - (\psi_j^{t}\, \psi_{tih} - \psi_i^{t}\, \psi_{tjh}) = \psi_i^{t}\, \nabla_h \psi_{tj} - \psi_j^{t}\, \nabla_h \psi_{ti} + u_j \nabla_i u_h$$

$$+ v_j \nabla_i v_h + e_j \nabla_i e_h - u_i \nabla_j u_h - v_i \nabla_j v_h - e_i \nabla_j e_h. \qquad \ldots(1.8)$$

Transvecting (1.8), with u^j, v^j, e^j, we get respectively

$$\{S_{jih} - (\psi_j^{t}\, \psi_{tih} - \psi_i^{t}\, \psi_{tjh})\}\, u^j = -\lambda^2 u_{ih} + \underset{u}{l}\, g_{ih} - u_i u^t\, \underset{u}{l}\, g_{th}$$

$$- e_i u^t e_{th} + \lambda \psi_i^{t}\, \underset{v}{l}\, g_{th} - v_{th}(v_i u^t + \lambda \psi_i^{t}) \qquad \ldots(1.9)$$

$$\{S_{jih} - (\psi_j^{t}\, \psi_{tih} - \psi_i^{t}\, \psi_{tjh})\}\, v^j = -\lambda^2 v_{ih} + \underset{v}{l}\, g_{ih} - v_i v^t\, \underset{v}{l}\, g_{th}$$

$$- e_i v^t e_{th} - \lambda \psi_i^{t}\, \underset{u}{l}\, g_{th} + (\lambda \psi_i^{t} - u_i v^t)\, u_{th} \qquad \ldots(1.10)$$

and

$$\{S_{jih} - (\psi_j^{t}\, \psi_{tih} - \psi_i^{t}\, \psi_{tjh})\}\, e^j = \underset{e}{l}\, g_{ih} - e_i e^t\, \underset{e}{l}\, g_{th}$$

$$- v_i e^t v_{th} - u_i e^t u_{th}. \qquad \ldots(1.11)$$

where $\underset{u}{l}$, $\underset{v}{l}$, $\underset{e}{l}$ denote the Lie derivative w.r.t. the vector field u, v, e respectively.

2. Quasi-normal $(\psi, g, u, v, e, \lambda)$-Structure

$(\psi, g, u, v, e, \lambda)$-structure is called quasi-normal if (Yano and Ki 1972)

$$S_{jih} - (\psi_j^{t}\, \psi_{tih} - \psi_i^{t}\, \psi_{tjh}) = 0. \qquad \ldots(2.1)$$

Using (2.1), eqns. (1.9) – (1.11) reduce to

$$\underset{u}{l}\, g_{ih} - u_i u^t \underset{u}{l}\, g_{th} + \lambda\psi_i^t \underset{v}{l}\, g_{th} - e_i u^t e_{th}$$
$$= \lambda^2 u_{ih} + v_{th}(v_i u^t + \lambda\psi_i^t) \quad \text{...(2.2)}$$

$$\underset{v}{l}\, g_{ih} - v_i v^t \underset{v}{l}\, g_{th} - \lambda\psi_i^t \underset{u}{l}\, g_{th} - e_i v^t e_{th}$$
$$= \lambda^2 v_{ih} - u_{th}(\lambda\psi_i^t - u_i v^t) \quad \text{...(2.3)}$$

and

$$\underset{e}{l}\, g_{ih} - e_i e^t \underset{e}{l}\, g_{th} - v_i e^t v_{th} - u_i e^t u_{th} = 0. \quad \text{...(2.4)}$$

Substituting (2.2) and (2.3) in (2.4) and on taking account of (1.1), we obtain

$$\lambda^2 \underset{e}{l}\, g_{ih} - \lambda^2 e_i e^t \underset{e}{l}\, g_{th} + v_i v^s e_{sh} - v_i e^t \underset{v}{l}\, g_{th} - u_i e^t \underset{u}{l}\, g_{th} + u_i u^s e_{sh} = 0. \quad \text{...(2.5)}$$

Transvecting (2.5) with u^i and using (1.1), we get

$$u^s e_{sh} = e^t \underset{u}{l}\, g_{th} - \frac{\lambda^2}{1-\lambda^2} u^i \underset{e}{l}\, g_{ih}. \quad \text{...(2.6)}$$

Substituting the value of $u^s e_{sh}$ in eqn. (2.5) and then transvecting the resulting equation with e^i and v^i respectively, we find

$$\underset{e}{l}\, g_{ih} = 0 \quad \text{...(2.7a)}$$

and

$$v^s e_{sh} = e^t \underset{v}{l}\, g_{th}. \quad \text{...(2.7b)}$$

From (1.1), (2.2) and (2.3), we have

$$\lambda^2(1-\lambda^2)\, v_{ih} + v_{sh}(u_i u^s - \lambda^2 v_i v^s - \lambda^2 e_i e^s) + \{\lambda(1-\lambda^2)\, \psi_i^t$$
$$- \lambda^2 v_i u^t - u_i v^t\} \underset{u}{l}\, g_{th} + \lambda^2 e_i e^t \underset{e}{l}\, g_{th} - \lambda^2 e_i v^t e_{th} = 0 \quad \text{...(2.8)}$$

which on transvecting with u^i and in view of (1.1) becomes

$$u^s v_{sh} = v^s \underset{u}{l}\, g_{sh} \quad (\lambda \neq \pm 1, \pm i) \quad \text{...(2.9)}$$

From (2.9), it can be seen that

$$(\underset{u}{l}\, g_{ij})\, u^j v^i = 0. \quad \text{...(2.10)}$$

Transvecting (2.2) and (2.3) with e^t, we get

$$e^i \underset{u}{l}\, g_{ih} - u^t e_{th} = \lambda^2 u_{ih} e^i \qquad \text{...(2.11}a\text{)}$$

and

$$e^i \underset{v}{l}\, g_{ih} - v^t e_{th} = \lambda^2 e^i v_{ih}. \qquad \text{...(2.11}b\text{)}$$

Transvecting (2.2) with v^i and also using (1.1) and (2.9), we find

$$u^t \underset{u}{l}\, g_{th} = v^t u_{hi}. \qquad \text{...(2.12)}$$

Making use of (2.6) and (2.7*a*, *b*) in (2.11*a*, *b*) we obtain

$$\text{(a)} \quad u_{ih} = 0, \qquad \text{(b)} \quad v_{ih} = 0, \qquad \text{(c)} \quad e_{ih} = 0. \qquad \text{...(2.13)}$$

Similarly from (2.9), (2.12) and (2.13), we have

$$\text{(a)} \quad \underset{u}{l}\, g_{sh} = 0, \quad \text{and} \quad \text{(b)} \quad \underset{v}{l}\, g_{sh} = 0. \qquad \text{...(2.14)}$$

Since $\underset{u}{l}\, g_{sh} = \nabla_s u_h + \nabla_h u_s = 0.$

which can be written as

$$\nabla_s u^s = 0.$$

Thus in a quasi-normal $(\psi, g, u, v, e, \lambda)$-structure, we get $u_{ih} = 0$, $v_{ih} = 0$, $e_{ih} = 0$ and $\nabla_i u^i = 0$, $\nabla_i v^i = 0$ and $\nabla_i e^i = 0$.

Now if we substitute the last set of equations in (1.9) – (1.11), we find

$$S_{jih} - (\psi^t_j \psi_{tih} - \psi^t_i \psi_{tjh}) = 0$$

which is the condition for a structure to be quasi-normal. Hence

Theorem 2.1 — If the $(\psi, g, u, v, e, \lambda)$-structure is quasi-normal, it is necessary and sufficient that U, V, E are harmonic vectors ($\lambda \neq \pm 1, \pm i$).

3 Invariant Submanifold of Co-dimension 2 of a Manifold with $(\psi, g, u, v, e, \lambda)$-Structure

We consider a submanifold V_m of V_n represented by

$$x^h = x^h(y^a) \qquad \text{...(3.1)}$$

and put

$$B^h_b\ \partial_b x^h, \ \partial_b = \partial/\partial y^b. \qquad \text{...(3.2)}$$

$a, b, c, \ldots$ run over $1, 2, \ldots, n$ and h, i, j run over 1 to m.

The induced Riemannian metric is given by

$$g_{cb} = g_{ji} B^j_c \, B^i_b \, . \qquad \ldots(3.3)$$

We denote by C^h_x , $2m - n$ mutually orthogonal unit normals to N. The equations of Gauss and those of Weingarten are given by (Yoshiko 1972)

$$\nabla_c \, B^h_b = \sum_x h_{cbx} C^h_x \qquad \ldots(3.4)$$

$$\nabla_c \, C^h_x = - h^a_{cx} \, B^h_a + \sum_y l_{cxy} C^h_x \qquad \ldots(3.5)$$

where

$$\nabla_c \, B^h_b = \partial_c B^h_b + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} B^j_c \, B^i_b - \begin{Bmatrix} a \\ c\,b \end{Bmatrix} B^h_a \qquad \ldots(3.6)$$

is the Vander Warden - Borotolotti covariant differential of B^h_b , $\begin{Bmatrix} a \\ c\,b \end{Bmatrix}$ being Christoffel symbols formed with g_{cb}.

$$\nabla_c \, C^h_x = \partial_c C^h_x + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} B^j_c \, C^i_x \, . \qquad \ldots(3.7)$$

h_{cbx} are the components of the second fundamental tensors w.r.t. the normals C^h_x .

$$h^a_{cx} = h_{cbx} g^{ba} \qquad \ldots(3.8)$$

g^{ba} being contravariant components of the induced metric and l_{cxy} components of the third fundamental tensor w.r.t. the normal C^h_x .

We assume that the submanifold V_m of V_n is ψ-invariant, hence we have (Yoshiko 1972, Yano and Okumura 1970)

$$\psi^h_i \, B^i_b = \psi^a_b \, B^h_a \, . \qquad \ldots(3.9)$$

ψ^a_b is a (1.1) tensor field of V_m. This shows that

$$\psi_{ih} B^i_b \, C^h_x = 0.$$

$\psi^h_i \, C^i_x$ is normal to the submanifold V_m. Thus

$$\psi^h_i \, C^i_x = \sum_y \gamma_{xy} C^h_y .$$

Since $\qquad \psi_{ih} C^i_x \, C^h_y = -\gamma_{xy}$

we see that

$$\gamma_{xy} = -\gamma_{yx}. \qquad ...(3.10)$$

Let

$$u^h = B^h_a u^a + \sum_x \alpha_x C^h_x \qquad ...(3.11)$$

$$v^h = B^h_a v^a + \sum_x \beta_x C^h_x \qquad ...(3.12)$$

and

$$e^h = B^h_a e^a + \sum_x \epsilon_x C^h_x \qquad ...(3.13)$$

u^a, v^a, e^a being vector fields of V_m and α_x, β_x, ϵ_x are functions in V_m.

Now from (1.1), (3.9) and (3.11) to (3.12), we obtain the following relations

$$\psi^a_c \psi^c_b = -\delta^a_b + u_b u^a + v_b v^a + e_b e^a \qquad ...(3.14)$$

$$u_b \alpha_x + v_b \beta_x + e_b \epsilon_x = 0 \qquad ...(3.15)$$

$$\psi^e_c \psi^d_b g_{ed} = g_{cb} - u_b u_c - v_b v_c - e_b e_c \qquad ...(3.16)$$

$$\alpha_x u^a + \beta_x v^a + \epsilon_x e^a = 0 \qquad ...(3.17)$$

$$\sum_y \gamma_{xy}\gamma_{yz} = -\delta_{xz} + \alpha_x\alpha_z + \beta_x\beta_z + \epsilon_x\epsilon_z \qquad ...(3.18)$$

$$\psi^h_a u^a = -\lambda v^h,\ \psi^h_a v^a = \lambda u^h,\ \psi^h_a e^a = 0 \qquad ...(3.19)$$

$$\left.\begin{aligned} &\sum_x \alpha_x\gamma_{xy} = -\lambda\beta_y,\ \sum_x \epsilon_x\gamma_{xy} = 0 \\ &\sum_x \beta_x\gamma_{xy} = \lambda\alpha_y \end{aligned}\right\} \qquad ...(3.20)$$

$$\left.\begin{aligned} u_a u^a &= 1 - \lambda^2 - \sum_x \alpha^2_x \\ v_a v^a &= 1 - \lambda^2 - \sum_x \beta^2_x \\ e_a e^a &= 1 - \sum_x \epsilon^2_x . \end{aligned}\right\} \qquad ...(3.21)$$

$$\left.\begin{aligned} u_a v^a &= -\sum_x \beta_x\alpha_x \\ u_a e^a &= -\sum_x \alpha_x\epsilon_x. \end{aligned}\right\} \qquad ...(3.22)$$

it can be easily seen that ψ_{cb} is skew-symmetric, since

$$\psi_{ji} B^j_c B^i_b = \psi_{cb}. \qquad ...(3.23)$$

Equations (3.14) – (3.22) show that a necessary and sufficient condition for ψ_b^a, g_{cb}, u_b, v_b, e_b and λ to admit a $(\psi, g, u, v, e, \lambda)$-structure is that $\sum_x \alpha_x^2 = 0$, $\sum_x \beta_x^2 = 0$ and $\sum_x \epsilon_x^2 = 0$. α_x, β_x and ϵ_x are zero or we say that u^h, v^h and e^h are always tangent to the submanifold. Since

$$(\nabla_j u_i - \nabla_i u_j)\, B_c^j\, B_b^i = \nabla_c u_b - \nabla_b u_c \qquad \text{...(3.24}a\text{)}$$

and

$$(\nabla_j u_i + \nabla_i u_j)\, B_c^j\, B_b^i = \nabla_c u_b + \nabla_b u_c \qquad \text{...(3.24}b\text{)}$$

hence

$$S_{ji}^h\, B_c^j\, B_b^i = (N_{cb}^a + u_{cb} u^a + v_{cb} v^a + e_{cb} e^a)\, B_a^h$$
$$+ \{\sum_x (\alpha_x u_{cb} + v_{cb}\beta_x + e_{cb}\epsilon_x)\}\, C_x^h\,. \qquad \text{...(3.25)}$$

where N_{cb}^a is the Nijenhuis tensor ψ_b^a.

This leads to the fact that if manifold with $(\psi, g, u, v, e, \lambda)$-structure is normal then the induced structure $(\psi, g, u, v, e, \lambda)$ on the submanifold V_m is also normal.

In view of (3.24*a*, *b*) we can also say that u^a, v^a, e^a will be harmonic vector fields on induced submanifold, if u^h, v^h and e^h are the harmonic vectors on the manifold V_n with $(\psi, g, u, v, e, \lambda)$-structure. Therefore

Theorem 3.1 — The induced $(\psi, g, u, v, e, \lambda)$-structure of the invariant submanifold V of co-dimension 2 with quasi-normal $(\psi, g, u, v, e, \lambda)$-structure is also quasi-normal.

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