Indian J. pure appl. Math., 11(5) : 640–647, May 1980

ON A SPACE OF ANALYTIC DIRICHLET TRANSFORMATIONS OF SEVERAL COMPLEX VARIABLES

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(*Received* 25 *November* 1978; *after revision* 5 *January* 1980)

In this paper topological structure of the vector space X of all analytic transformations, represented by Dirichlet series in two complex variables, in a polystrip has been studied. Two equivalent topologies have been considered on X and it has been shown that X, equipped with either of the two topologies, becomes a Fréchet space which is non-normable. After introducing proper bases in X, conditions have been obtained such that a base in X become a proper base. Also it has been proved that if X_1 and X_2 be two closed subspaces of X with their respective proper bases $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ then there exists a linear homeomorphism T from X_1 onto X_2 such that $T\alpha_{mn} = \beta_{mn}$ and conversely. Also certain linear transformations from X into a subspace Y of it, equipped with some Fréchet topology have been characterized.

1. INTRODUCTION

A topological study of the space of entire functions has an important place in the theory of distributions, general problems in the theory of approximations and expansions of functions in series (see for instance Gelfand and Shilov (1964). Also the modern proof of Riemann's celebrated theorem on conformal mapping is derived on account of the typical structure of such a space (see Ahlfors 1966, chap. 6). Keeping this in view it becomes essential to study in as much details as possible, the spaces of Dirichlet transformations. In this direction Kamthan (1969) initiated the study of spaces of entire Dirichlet transformations of one complex variable. This was further carried on by Hussain and Kamthan (1968) and by Kamthan and Gautam (1972, 1975a, b, 1976). After this one is naturally interested in a study of spaces of entire (analytic) Dirichlet transformations of several complex variables. This, however, is not in general simple. But a recent study of the spaces of analytic (entire) functions given by Taylor series of several complex variables (see Kamthan 1973a, b; Kamthan and Gupta 1974, 1975, 1976) and a characterization of analytic Dirichlet transformations of several complex variables in terms of its coefficients (see Artemiades 1953, Gromov 1969, Kardas and Culyk 1972) has motivated us to look into the problem.

2. Notations and Terminologies

Let X denote the class of all analytic transformations* $f : C^2 \to C$ in

$$P(\Delta_1, \Delta_2) = \{(s_1, s_2) : \text{Re}\,(s_j) = \sigma_j < \Delta_j,\ j = 1, 2\}$$

where $-\infty < \Delta_1, \Delta_2 < \infty$ are arbitrary fixed real numbers such that

$$f(s_1, s_2) = \sum_{m,\,n=1}^{\infty} a_{mn} e^{s_1 \lambda_m + s_2 \mu_n};\ (s_1, s_2) \in P(\Delta_1, \Delta_2) \qquad \ldots(2.1)$$

with

$$\left.\begin{array}{ll} 0 < \lambda_1 < \lambda_2 \ldots < \lambda_m, & \lim\limits_{m \to \infty} \lambda_m = \infty \\ 0 < \mu_1 < \mu_2 \ldots < \mu_n, & \lim\limits_{n \to \infty} \mu_n = \infty \end{array}\right\} \qquad \ldots(2.2)$$

and for different $f \in X$ only a_{mn}'s change. We know that (Artemiades 1953, Gromov 1969, Kardaa and Culyk 1972) each such f is characterized as follows:

$$\limsup_{m,n \to \infty} \frac{\log |a_{mn}| + \Delta_1 \lambda_m + \Delta_2 \mu_n}{\lambda_m + \mu_n} \leqslant 0. \qquad \ldots(2.3)$$

It can be easily verified that with usual vector addition and scalar multiplication X forms a vector space over C.

3. Topological Structure of X

For $-\infty < \sigma_j < \Delta_j$, $(j = 1, 2)$** we define

$$M(f; \sigma_1, \sigma_2) = \sup \{ |f(\sigma_1 + it_1, \sigma_2 + it_2)| : -\infty < t_1, t_2 < \infty\}.$$

Then, $M(\ldots; \sigma_1, \sigma_2)$ defines a semi-norm (in fact a norm) on X. Thus, the sequence $\{M(\ldots; \sigma_{1,k}, \sigma_{2,k}) : -\infty < \sigma_{j,k} < \Delta_j, k = 1, 2, \ldots\}$ with

$$\sigma_{j,1} < \sigma_{j,2} < \ldots < \sigma_{j,k} \to \Delta_j,$$

as $k \to \infty$, $j = 1, 2$ defines a non-decreasing family of semi-norms on X. Let τ_1 be the locally convex Hausdorff topology on X generated by this family of semi-norms. This topology is also given by the invariant metric d, defined by

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{M(f - g; \sigma_{1,k}, \sigma_{2,k})}{1 + M(f - g; \sigma_{1,k}, \sigma_{2,k})}, \text{ (Fréchet combination).}$$

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*We consider the case of two variables only for the sake of simplicity although our results are true for any finite number of variables.

**Throughout, by σ_j we shall mean a real number such that $\sigma_j < \Delta_j$ $(j = 1, 2)$ unless otherwise specified.

Next, for each $f \in X$, define

$$p(f;\ \sigma_1, \sigma_2) = \sum_{m,n=1}^{\infty} |\ a_{mn}\ | \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n).$$

Then, $\{p(\ldots;\ \sigma_{1,k}, \sigma_{2,k}) : -\infty < \sigma_{j,k} < \Delta_j,\ j = 1, 2\}$ defines a family of semi-norms on X. Let τ_2 be the locally convex Hausdorff topology generated by this family. This topology is also given by

$$\lambda(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p(f - g;\ \sigma_{1,k}, \sigma_{2,k})}{1 + p(f - g;\ \sigma_{1,k}, \sigma_{2,k})}.$$

Invoking the Cauchy-Ritt inequality for several complex variables (cf. Jain and Gupta 1976) namely

$$|\ a_{mn}\ | \leqslant M(f;\ \sigma_1, \sigma_2) \exp(-\ \sigma_1 \lambda_m - \sigma_2 \mu_n)$$

it follows that, for $k_1, k_2 > 0$, there exists a constant $K(k_1, k_2)$ such that

$$M(f;\ \sigma_1, \sigma_2) \leqslant p(f;\ \sigma_1, \sigma_2) \leqslant K(k_1, k_2)\ M(f;\ \sigma_1 + k_1, \sigma_2 + k_2)$$

where $\sigma_j + k_j < \Delta_j$, $j = 1, 2$. The above inequalities yield that the topologies τ_1 and τ_2 are equivalent. We may mention here that we may interchange the roles of τ_1 and τ_2 according to convenience in our analysis. Hereafter, by X we shall mean the linear space X equipped with the topology τ_1 or τ_2.

Theorem 1 — The space is a Fréchet space.

PROOF : Let $\{f_p\}$ be a Cauchy sequence in χ, where

$$f_p(s_1, s_2) = \sum_{m,\,n=1}^{\infty} a_{mn}^{(p)} \exp(s_1 \lambda_m + s_2 \mu_n).$$

It follows that $\{f_p\}$ is a Cauchy sequence with respect to each $M(\ldots;\ \sigma_{1,k}, \sigma_{2,k})$, $k \geqslant 1$. This in view of Cauchy-Ritt inequality for several complex variables, gives

$$|\ a_{mn}^{(p)} - a_{mn}^{(q)}\ | < \epsilon \exp(-\ \lambda_m \sigma_{1,k} - \mu_n \sigma_{2,k}) \qquad \ldots(3.1)$$

for all $p, q \geqslant N_0$ and $m, n = 1, 2, 3, \ldots$. The proof now follows on the usual lines.

Theorem 2 — The space χ is not a normed space.

PROOF : We know that a space is not a normed space if it is not locally bounded (see Wehausen 1938). Let us consider an arbitrary neighbourhood G of zero in χ. Then we can find $\sigma_{1,k}$ and $\sigma_{2,k}$ such that for some $\epsilon > 0$,

$$S(\sigma_{1,k}, \sigma_{2,k};\ \epsilon) = \{f : f \in \chi,\ M(f;\ \sigma_{1,k}, \sigma_{2,k}) < \epsilon\} \subseteq G.$$

Define $f_p \in \chi$ as

$$f_p(s_1, s_2) = \frac{\epsilon}{2} \cdot \frac{\exp(s_1\lambda_p + s_2\mu_p)}{\exp(\lambda_p\sigma_{1,k} + \mu_p\sigma_{2,k})}, \; p = 1, 2, 3, \ldots .$$

Choose a sequence $\{\epsilon_p\}$ of scalars, where

$$\epsilon_p = \exp(-\lambda_p\sigma_{1,k} - \mu_p\sigma_{2,k}).$$

Clearly $\{f_p\} \subset G$ and $\epsilon_p . f_p \not\to 0$ and therefore, G is unbounded. The arbitrariness of G proves the result.

4. Bases in χ

A sequence $\{f_{mn}\}$ in χ is said to be a basis in χ if to every $f \in \chi$, there corresponds a unique sequence $\{a_{mn}\}$ of scalars such that $f = \Sigma\, a_{mn} f_{mn}$, the convergence of the infinite series being with respect to the topology on χ. Denoting the coefficient functionals by $\varphi_{mn}(f)$, we have $f = \Sigma\, \varphi_{mn}(f) . f_{mn}$. The basis $\{f_{mn}\}$ will be called a Schauder basis if each such φ_{mn} is continuous with respect to the topology on χ.

Trivially $\{\delta_{mn}\}$, $\delta_{mn} = \exp(s_1\lambda_m + s_2\mu_n)$, is basis and furthermore a Schauder basis χ. (see Newns 1953, p. 431–32). We now formulate the following:

Definition — A basis $\{\alpha_{mn}\}$ in X is said to be proper if for all sequences $\{a_{mn}\}$ of scalars, we have

$$\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn} \text{ converges} \Leftrightarrow \sum_{m,n=1}^{\infty} a_{mn}\delta_{mn} \text{ converges.}$$

Theorem 3 — Let $\{\alpha_{mn}\}$ be a sequence in X. Then the following conditions are equivalent :

$$(\alpha) \quad \limsup_{m,n\to\infty} \left[\frac{\log M(\alpha_{mn}; \sigma_1, \sigma_2) - \Delta_1\lambda_m - \Delta_2\mu_n}{\lambda_m + \mu_n}\right] < 0, \; \forall\, \sigma_j.$$

$$(\mathrm{A}_1) \quad \limsup_{m,n\to\infty} \frac{\log|a_{mn}| + \Delta_1\lambda_m + \Delta_2\mu_n}{\lambda_m + \mu_n} \leqslant 0$$

$$\Rightarrow \sum_{m,n=1}^{\infty} M(a_{mn}\alpha_{mn}; \sigma_1, \sigma_2) \text{ converges for all } \sigma_j.$$

$$(\mathrm{A}_2) \quad \limsup_{m,n\to\infty} \frac{\log|a_{mn}| + \Delta_1\lambda_m + \Delta_2\mu_n}{\lambda_m + \mu_n} \leqslant 0$$

$$\Rightarrow \sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn} \text{ converges in } X.$$

$$(\mathrm{A}_3) \quad \limsup_{m,n\to\infty} \frac{\log|a_{mn}| + \Delta_1\lambda_m + \Delta_2\mu_n}{\lambda_m + \mu_n} \leqslant 0$$

$$\Rightarrow a_{mn}\alpha_{mn} \to 0 \text{ in } X.$$

Theorem 4 — Let $\{\alpha_{mn}\}$ be a sequence in χ. Then the following conditions are equivalent:

$$(\beta) \qquad \lim_{\sigma_j \to \Delta_j} \left\{ \liminf_{m,n \to \infty} \frac{\log M(\alpha_{mn}; \sigma_1, \sigma_2) - \Delta_1 \lambda_m - \Delta_2 \mu_n}{\lambda_m + \mu_n} \right\} \geqslant 0$$

(B_1) $\sum_{m,n=1}^{\infty} M(a_{mn}\alpha_{mn}; \sigma_1, \sigma_2)$ converges for all σ_j

$$\Rightarrow \limsup_{m,n \to \infty} \frac{\log |a_{mn}| + \Delta_1 \lambda_m + \Delta_2 \mu_n}{\lambda_m + \mu_n} \leqslant 0.$$

(B_2) $\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$ converges in X

$$\Rightarrow \limsup_{m,n \to \infty} \frac{\log |a_{mn}| + \Delta_1 \lambda_m + \Delta_2 \mu_n}{\lambda_m + \mu_n} \leqslant 0$$

(B_3) $a_{mn}\alpha_{mn} \to 0$ as $m, n \to \infty$ in X

$$\Rightarrow \limsup_{m,n \to \infty} \frac{\log |a_{mn}| + \Delta_1 \lambda_m + \Delta_2 \mu_n}{\lambda_m + \mu_n} \leqslant 0.$$

PROOFS OF THEOREMS 3 AND 4 : These follow on the lines similar to those in the proofs of Lemmas 4.1 and 4.2 of Kamthan and Gupta (1974), respectively. The details are omitted.

Theorems 3 and 4 when combined together give the following:

Theorem 5 — A basis $\{\alpha_{mn}\}$ in χ is proper if and only if (α) and (β) are satisfied.

5. LINEAR OPERATORS ON χ

Theorem 6 — Let $\{\alpha_{mn}\}$ be a sequence of functions in χ. Then there exists a continuous linear operator T on χ such that $T\delta_{mn} = \alpha_{mn}$ $(m, n = 1, 2, 3, \ldots)$ if and only if

$$\limsup_{m,n \to \infty} \frac{\log M(\alpha_{mn}; \sigma_1, \sigma_2) - \Delta_1 \lambda_m - \Delta_2 \mu_n}{\lambda_m + \mu_n} < 0 \qquad \ldots(5.1)$$

for all $\sigma_j < \Delta_j$, $j = 1, 2$.

PROOF : Let $\sigma_j < \Delta_j$, $j = 1, 2$ be given. Then, there exist $\rho_j < \Delta_j$, $j = 1, 2$ such that T maps $\chi(\rho_1, \rho_2)$ continuously into $\chi(\sigma_1, \sigma_2)$. Thus there exists a constant K such that $p(\alpha_{mn}; \sigma_1, \sigma_2) \leqslant K \exp(\rho_1 \lambda_m + \rho_2 \mu_n)$. Choosing

$$\Delta = \max(\rho_1 - \Delta_1, \rho_2 - \Delta_2)$$

this inequality implies

$$\limsup_{m,n\to\infty} \frac{\log p(\alpha_{mn};\, \sigma_1, \sigma_2) - \Delta_1\lambda_m - \Delta_2\mu_n}{\lambda_m + \mu_n} \leqslant \Delta < 0.$$

Since $M(\alpha_{mn};\, \sigma_1, \sigma_2) \leqslant p(\alpha_{mn};\, \sigma_1, \sigma_2)$, it follows that (5.1) holds.

To prove the converse, we note that each $f \in \chi$ can uniquely be expressed as $f = \Sigma\, a_{mn}\delta_{mn}$, where the series on the right converges in the topology of χ. Let us define a linear operator T on χ by $T(f) = T(\Sigma\, a_{mn}\delta_{mn}) = \Sigma\, a_{mn}\alpha_{mn}$. Then T is well defined since the series $\Sigma\, a_{mn}\alpha_{mn}$ converges in χ (Theorem 3). The continuity follows on the lines of the proof of Theorem 5.1 of Kamthan and Gupta (1974).

Theorem 7 — If T is a linear homeomorphism of χ into itself, then $\{T\delta_{mn}\}$ is a proper basis in some closed subspace χ_0 of χ. Conversely, if $\{\alpha_{mn}\}$ is a proper basis in a closed subspace χ_0 of χ, then there exists a linear homeomorphism T of χ onto χ_0 such that $T\delta_{mn} = \alpha_{mn};\, m, n = 1, 2, 3, \ldots$.

PROOF: The proof of the theorem is omitted, since it is a direct consequence of Theorem 5 of Banach (1932, p. 41).

Corollary — Let X_1 and X_2 be two closed subspaces of X. If $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are, respectively, proper bases for X_1 and X_2, then there exists a linear homeomorphism T from X_1 onto X_2 such that $T\alpha_{mn} = \beta_{mn}\ (m, n = 1, 2, \ldots)$. Conversely, if T is a linear homeomorphic map from X_1 onto X_2 and $\{\alpha_{mn}\}$ is a proper basis for X_1, then the sequence $\{T\alpha_{mn}\}$ is a proper basis for X_2.

Remark: If $\{\alpha_{mn}\}$ is a proper basis in X, then in view of the fact that there exists a linear homeomorphism on X mapping δ_{mn} onto α_{mn}, there is no loss of generality in taking δ_{mn} as α_{mn} for all $m, n = 1, 2, \ldots$. Thus the result in Theorem 1 can be restated as follows:

Theorem 8 — Let $\{\alpha_{mn}\}$ be proper basis in X and $\{\varphi_{mn}\}$ be any sequence in X. Then there exists a linear operator P on X such that $P\alpha_{mn} = \varphi_{mn}\ (m, n = 1, 2, 3, \ldots)$ if and only if $\{\varphi_{mn}\}$ satisfies (5.1).

Let us consider another space $(Y, \mathcal{T})$, where Y is a subspace of X and $\mathcal{T}$ is any Fréchet topology on Y. We assume that $\{\, \|\ldots\|_\nu : \nu = 1, 2, 3, \ldots\}$ stands for the family of semi-norms on Y, which generates the topology $\mathcal{T}$. Now, we are interested in knowing when P is a continuous linear map from (X, τ) into $(Y, \mathcal{T})$. In this direction we have the following result:

Theorem 9 — Let $\{\alpha_{mn}\}$ be a proper basis in χ. Then a necessary and sufficient condition for the operator P to be a continuous linear map from (χ, τ) into $(Y, \mathcal{T})$ is that all $\varphi_{mn} \in Y$ and

$$\limsup_{m,n\to\infty} \frac{\log \| \varphi_{mn} \|_\nu - \triangle_1\lambda_m - \triangle_2\mu_n}{\lambda_m + \mu_n} < 0, \quad \nu = 1, 2, \ldots \qquad \ldots(5.2)$$

a fortiori, the series $\Sigma\, a_{mn}\varphi_{mn}$ converges in Y for each $f \in X$.

PROOF : The proof of the necessary part is simple and that of the sufficiency part can easily be derived by using an argument similar to that used in the proof of Theorem 6.

In particular, P is continuous if the topology on Y is taken to be as the induced topology from τ, and $\{\varphi_{mn}\}$ satisfies the condition (5.1) in place of (5.2). Since the induced topology is weaker than the topology determined by the supnorm on Y, it is a natural question to know whether P preserves continuity or not in case the topology on Y is taken to be any Fréchet topology weaker than the supnorm topology on Y and $\{\varphi_{mn}\}$ satisfies the condition (5.1). The answer will be in the affirmative if we restrict the class Y suitably. In fact we prove the following:

Theorem 10 — Suppose that Y consists of all functions $f \in X$ for which

$$\sup_{\sigma_1<\triangle_1,\, \sigma_2<\triangle_2} \{M(f;\, \sigma_1, \sigma_2)\} < \infty$$

and that the topology on Y is weaker than that determined by the supnorm topology on Y. Let $\varphi_{mn} \in Y$ be uniformly continuous functions in $P(\triangle_1, \triangle_2)$ satisfying the condition

$$\limsup_{m,n\to\infty} \frac{\log M(\varphi_{mn};\, \sigma_1, \sigma_2) - \triangle_1\lambda_m - \triangle_2\mu_n}{\lambda_m + \mu_n} < 0.$$

Then P is a continuous linear map from X to Y.

PROOF : It follows by using arguments similar to those used in the proof of Theorem 4.6 of Gupta (1973).

ACKNOWLEDGEMENT

The authors wish to acknowledge their indebtedness and express their thanks to Dr. P. K. Jain for his guidance and encouragement in the preparation of this paper. They are also deeply grateful to the referee for his very useful comments and suggestions towards the improvement of the paper.

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