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ON TWO CAPACITIES IN A HARMONIC SPACE

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Let e be a compact set and B be an open ball in $\mathbb{R}^2$ such that $e \subset B \subset \mathbb{R}^2$. With respect to B, we define the Green capacity of e; also we have the notion of the logarithmic capacity of e in $\mathbb{R}^2$. In this article we investigate the relation between these two capacities. Actually we consider this problem in a more general set up of a harmonic space Ω without positive potentials (examples of such a space being $\mathbb{R}^1$, $\mathbb{R}^2$ and the parabolic Riemann surfaces) and with respect to each compact set e in Ω we introduce two set functions $N(e)$ and $\varphi(e)$ resembling the Green capacity and the logarithmic capacity and prove that locally there exist two positive constants a and b independent of the compact sets e satisfying the conditions $a\,\varphi(e) \leqslant N(e) \leqslant b\,\varphi(e)$.

1. INTRODUCTION

Let K be a compact set in $\mathbf{R}^2$. Let B be a ball containing K. Then there exists in B a Green potential p with its associated measure $\mu \geqslant 0$ supported by K such that $p \leqslant 1$ in B; and p is the greatest function in B satisfying the above conditions. The value $\mu(K)$ is known as the 'Green capacity' of K with respect to B.

Now for any measure μ in $\mathbf{R}^2$ with compact support, the 'energy integral' of μ is $I(\mu) = \iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y)$. This is well-defined and satisfies the inequality $-\infty < I(\mu) \leqslant \infty$. For the compact set K, if $V_K = \inf\limits_{\mu \in M} I(\mu)$ where M is the family of positive measures of total mass 1 supported by K, then the 'logarithmic capacity' of K is $\exp(-V_K)$.

Thus for the same compact set K in $\mathbf{R}^2$ we define two important capacities. In this article we are concerned with the interesting problem of determining the relation between these two capacities.

Actually in a harmonic space Ω without positive potentials, let ω be a domain in which is defined a potential > 0. We introduce two set functions $N(e)$ and $\varphi(e)$ on the class of compact sets e of ω resembling the Green capacity and the logarithmic capacity and prove that locally $a\,\varphi(e) \leqslant N(e) \leqslant b\,\varphi(e)$ where a and b are two positive finite numbers independent of the compact sets e.

2. The N-Capacity in a B.P. Space

Let Ω be a locally compact, not compact, connected and locally connected space provided with a sheaf of harmonic functions satisfying the axioms 1, 2, 3 of Brelot (1945); we assume that the constants are harmonic functions in Ω. Then Ω is called a 'B.H. space'.

A domain ω in a B.H. space Ω is a 'P-domain' if there exists a potential > 0 in ω. Ω is a 'B.P. space' or a 'B.S. space' depending on whether Ω is a P-domain or not. We remark that a hyperbolic Riemann surface and $\mathbf{R}^n$ $(n \geqslant 3)$ are examples of B.P. spaces while a parabolic Riemann surface and $\mathbf{R}^n$ $(n = 1, 2)$ are examples of B.S. spaces.

'Let us suppose in this section that Ω is a B.P. space with a countable base'.

Let (ω, z) be a fixed pair where ω is a relatively compact open set in Ω and z a point in ω. Then for any positive superharmonic function v in Ω, one can associate with v a Radon measure $\mu_v^{\omega, z} \geqslant 0$ on $\bar{\Omega}$ representing it. (For the characterizations of the measure, see section 15 of Hervé 1962).

Also if v is a potential in Ω and $\mathfrak{a}$, the point at infinity, $\mu_v^{\omega, z}(\mathfrak{a}) = 0$ and $\int d\mu_v^{\omega, z} = \int v \, d\rho_z^{\omega}$. Consequently if u and v are two potentials such that $u \geqslant v$ then $\int d\mu_u^{\omega, z} \geqslant \int d\mu_v^{\omega, z}$.

Definition 1 — Let e be a compact set in Ω and $v = \hat{R}_1^e$. Then $N(e) = \int d\mu_v^{\omega, z}$, the total measure associated with v, is called the N-capacity of e.

Theorem 2 — Let Ω be a B.P. space with a countable base satisfying the axiom D. Then on the class of compact sets e, $N(e)$ is a strong capacity, that is, it is an increasing function which is right continuous and strongly subadditive.

Proof: If e is a compact set $N(e) = \int \hat{R}_1^e \, d\rho_z^{\omega}$. Since R_1^e differs from $\hat{R}_1^e$ at most only on a polar set, we have $N(e) = \int R_1^e \, d\rho_z^{\omega}$.

Then using Theorem 26.1 of Hervé (1962), one completes the proof of the theorem.

Suppose now that the sheaf satisfies the 'local proportionality' in Ω; that is, for any domain ω and a point $x \in \omega$ if p and q are two potentials > 0 in ω with support x, then p and q are proportional.

Integral representation — Let Ω be a B.P. space with a countable base satisfying the local axiom of proportionality.

Then for a given $y \in \Omega$, one can choose a potential $p_y(x)$ with support y such that $p_y(x)$ is a continuous function of y in $\Omega - \{x\}$. Further (Theorem 18.2 of Hervé 1962) any potential $u(x)$ in Ω has a unique integral representation of the form

$$u(x) = \int p_y(x)\, d\underset{u}{\lambda}(y)$$

where $\underset{u}{\lambda}$ is a measure $\geqslant 0$ in Ω such that

$$d\underset{u}{\lambda}(y) = \frac{d\underset{u}{\mu}^{\omega,z}(y)}{\int p_y\, d\rho_z^{\omega}} \text{ where } \underset{u}{\mu}^{\omega,z}$$

is the measure associated with u as above with respect to the fixed pair (ω, z).

Moreover, $\int p_y\, d\rho_z^{\omega}$ as a function of y is a lower semi-continuous function which is locally bounded in Ω (Hervé 1962, p. 481). Then we have the following proposition.

Proposition 3 — Let K be a fixed compact set in a B.P. space Ω with a countable base satisfying the local axiom of proportionality. Let u and v be two potentials in Ω with support in K such that $u \leqslant v$. Then there exists a constant δ depending only on K such that $\int d\underset{u}{\lambda} \leqslant \delta \int d\underset{v}{\lambda}$.

PROOF : We know that $\underset{u}{\mu}^{\omega,z}$ and $\underset{v}{\mu}^{\omega,z}$ have support in K and $\int d\underset{u}{\mu}^{\omega,z} \leqslant \int d\underset{v}{\mu}^{\omega,z}$.

Since $\int p_y\, d\rho_z^{\omega}$ is bounded locally in Ω, there exists a constant $\delta > 0$ such that $\delta^{-1/2} \leqslant \int p_y\, d\rho_z^{\omega} \leqslant \delta^{1/2}$ for $y \in K$.

Hence

$$\begin{aligned} \int d\underset{u}{\lambda} &\leqslant \delta^{1/2} \int d\underset{u}{\mu}^{\omega,z}(y) \\ &\leqslant \delta^{1/2} \int d\underset{v}{\mu}^{\omega,z}(y) \\ &\leqslant \delta \int d\underset{v}{\lambda}. \end{aligned}$$

The proposition is proved.

Theorem 4 — Let Ω be a B.P. space with a countable base satisfying the local axiom of proportionality. Let K be a fixed compact set in Ω. Then for any compact set e in K.

$$\alpha N(e) \leqslant c(e) \leqslant \beta N(e)$$

where $c(e)$ is the total measure $\int_{\nu} d\lambda$ associated with $\nu = \hat{R}_1^e$ in an integral representation as defined above, and α and β are two constants depending only on K.

PROOF: Now $\nu(x) = \int_{\nu} p_y(x)\, d\lambda(y)$.

Since $d\lambda_{\nu}(y) = \dfrac{d\mu_{\nu}^{\omega, z}(y)}{\int p_y\, d\rho_z^{\omega}}$ and $\int p_y\, d\rho_z^{\omega}$ is bounded on K, there exist constants α and $\beta > 0$ not depending on the variable compact set $e \subset K$ satisfying the inequalities

$$\alpha \int_{\nu} d\mu^{\omega, z} \leqslant \int_{\nu} d\lambda \leqslant \beta \int_{\nu} d\mu^{\omega, z}.$$

Hence $\alpha N(e) \leqslant c(e) \leqslant \beta N(e)$.

3. THE φ-CAPACITY IN A B.S. SPACE

Let Ω be a B.S. space. We fix an outer-regular compact set K and a non-constant harmonic function $H \geqslant 0$ in $(\Omega - K)$ tending to zero on ∂K. Flux at infinity of H is taken to be 1.

Let $\mathcal{U}$ be a fixed ultrafilter finer than the filter of sections of relatively compact open sets in Ω.

Given a function u we denote $D(u) = \lim_{\mathcal{U}} \bar{H}_u^{\omega}$ when the limit exists according to the ultrafilter $\mathcal{U}$, where $\bar{H}_u^{\omega}$ stands for the upper Dirichlet solution in the relatively compact open set ω.

A superharmonic function u in a B.S. space Ω is said to be 'BH_e-potential' (or simply a 'BH-potential') if for some α, $D(u - \alpha H) \equiv 0$.

Suppose now that Ω is a B.S. space with a countable base satisfying locally axiom D. Then given any non-locally polar compact set e in Ω, there exists a unique BH-potential q_e in Ω (called the 'BH-capacitary potential' of e) with total flux at infinity -1, having support in e, such that for some constant λ, $q_e = \lambda$ q.e. (that is except possibly on a locally polar set) on e and $q_e \leqslant \lambda$ on Ω. [This is proved under some restrictive conditions by Anandam (1976). But it is not difficult to prove this result here modifying the proof of Theorem 3.2 of Anandam (1976)].

We say that the 'φ-capacity' of e is $\varphi(e) = 1/\lambda$ when e is a compact, non-locally-polar set and $\varphi(e) = 0$ when e is a locally polar compact set.

A domain X in Ω is said to be a 'positive set' if any BH-capacitary potential with compact support in X is positive in X. For example, any domain in $\mathbf{R}^2$ of diameter not greater than 1 is a positive set.

It is known that on the class of compact sets contained in a positive set, the set function φ defines a weak capacity, that is, φ is increasing, right continuous and subaddive [see the Remark on p. 355 of Anandam (1976)].

Now note that any positive set X is necessarily a P-domain. Hence considering X as a B.P. space, on the class of compact sets in X we can define the strong capacity $N(e)$ as in the previous section. The relation between $\varphi(e)$ and $N(e)$ when e is a variable compact set contained in a positive set X is given in the following theorem. [For an analogue of this theorem in the classical case, see Brelot (1945)].

Theorem 5 — Let X be a positive set in a B.S. harmonic space Ω with a countable base satisfying locally the axiom D and the axiom of proportionality. Let K be a fixed compact set in X. Then $a\varphi(e) \leqslant N(e) \leqslant b\varphi(e)$ where a and b are two positive constants depending only on K and e is a variable compact set in K.

PROOF: First we note that

$\{e : \varphi(e) = 0\} = \{e : N(e) = 0\} = \{$the family of locally polar compact sets in $X\}$.

Hence we shall consider only the case when the compact set e is not locally polar.

Secondly, X being a positive set, for any compact $e \subset X$, $\varphi(e) > 0$.

Thirdly, we take K to be non-polar; otherwise the theorem is trivial.

Now given $y \in \Omega$, let $q_y(x)$ be the unique BH-potential in Ω, with support y and flux at infinity -1. Here $q_y(x)$ is a continuous function of y in $\Omega - \{x\}$ (Guessous 1977).

Fix a pair (ω, z) where $\bar{\omega} \subset X$.

Then $p_y(x) = q_y(x) - q_y^X(x)$ in X, where $q_y^X(x)$ is the greatest harmonic minorant of $q_y(x)$ in X, is continuous in y if $x \neq y$ [see the proof of Theorem 2.4 of Anandam (1976)]. Hence we have an integral representation of a potential in X by means of the function $p_y(x)$.

If $p > 0$ is a potential in X, let us denote by $m(p)$ the total measure associated with p in an integral representation by means of the function $p_y(x)$ and the fixed pair (ω, z).

For a non-locally-polar compact set e in X [see Anandam (1976, Theorem 2.4) and Guessous (1977)] we have for $x \in \Omega$

$$\varphi(e)\, q_e(x) = \int q_y(x)\, d\lambda_e(y).$$

Here $\underset{e}{\lambda} \geqslant 0$ is the measure associated with the potential $\varphi(e)\, q_e - h_e$ in X where h_e is the greatest harmonic minorant of $\varphi(e)\, q_e$ in X. Thus

$$m(\varphi(e)\, q_e - h_e) = \int d\underset{e}{\lambda}$$

$$= -\text{flux of } \varphi(e)\, q_e \text{ at infinity [Anandam (1976), p. 346)]}$$

$$= \varphi(e).$$

Since $\varphi(e)\, q_e - h_e$ is a potential in X such that

$$0 < \varphi(e)\, q_e - h_e \leqslant 1 \text{ on } e,\ \varphi(e)\, q_e - h_e \leqslant (\hat{R}_1^e)_X \text{ in } X \text{ (axiom D)}.$$

Hence for any compact set e contained in K, there exists $\delta > 0$ depending only on K (Proposition 3) such that

$$m(\varphi(e)\, q_e - h_e) \leqslant \delta m\, ((\hat{R}_1^e)_X).$$

That is, $\varphi(e) \leqslant \delta\, c(e)$ (definition of $c(e)$ as in Theorem 4)

$$\leqslant \delta\beta\, N(e) \text{ (Theorem 4).} \qquad \text{...(1)}$$

Let e be a compact set in K such that $\varphi(e)\, q_e \geqslant A$ on K where $0 < A < 1$.

Then by Lemma 3.6 of Anandam (1976), $\dfrac{1}{\varphi(k)} \geqslant \dfrac{A}{\varphi(e)}$.

Hence $\quad \varphi(e) \geqslant A\varphi(K)$

$$= \lambda c(K) \text{ where } \lambda \text{ is a constant}$$

$$\geqslant \lambda\alpha\, N(K) \quad \text{(Theorem 4)}$$

$$\geqslant \lambda\alpha\, N(e). \qquad \text{...(2)}$$

Let now e be a compact set in K for which $\varphi(e)\, q_e < A$ atleast at one point of K.

Then h_e, the greatest harmonic minorant of $\varphi(e)\, q_e$ in X, is less than A atleast at some point of K.

From the fact that $\sup\limits_{e \subset K}^{e} h_e \leqslant B \inf\limits_{e \subset K}^{e} h_e$ where B is a constant independent of e, we deduce that $\sup\limits_{e \subset K}^{e} h_e \leqslant \eta$ where η should be less than 1 since each h_e is less than 1 at some point of K. (Otherwise a subsequence of (h_e) will converge to 1 uniformly on K, a contradiction).

Hence for such a compact set e in K,

$$\varphi(e)\, q_e - h_e \geqslant (1 - \eta) \text{ q.e. on } e \text{ and hence}$$

$$\varphi(e)\, q_e - h_e \geqslant (1 - \eta)\, (\hat{R}_1^e\,)x \text{ in } X.$$

By Proposition 3, then,

$$\delta\varphi(e) \geqslant (1 - \eta)\, c(e)$$

$$\geqslant (1 - \eta)\, \alpha N(e). \qquad \ldots(3)$$

From (2) and (3) we obtain for all compact sets e in K, $b\, \varphi(e) \geqslant N(e)$ where b is a constant depending only on K.

Combining this with (1), we conclude that there exist positive constants a and b depending only on K such that for any compact e in K,

$$a\varphi(e) \leqslant N(e) \leqslant b\varphi(e).$$

This completes the proof of the theorem.

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