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# ON ABSOLUTE NÖRLUND SUMMABILITY FACTOR OF LEGENDRE SERIES

SUSHIL SHARMA

*Department of Mathematics, Madhav Vigyan Mahavidyalaya, Ujjain* 456001 (*M.P.*)

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In the present note we improve a theorem, established by Beohar (1969) on absolute Nörlund summability factor of Legendre series.

## 1. DEFINITIONS

Let $\{s_n\}$ be the sequence of partial sums of a series $\Sigma\, u_n$. Let $t_n$ be the Nörlund mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$, real or complex.

Let us write

$$P_n = p_0 + p_1 + p_2 + \dots + p_n, \quad P_n \neq 0,\, P_{-1} = p_{-1} = 0.$$

The series $\Sigma\, u_n$ is said to be absolutely summable by Nörlund means, if the series

$$\Sigma \mid t_n - t_{n-1} \mid$$

is convergent.

In the special case when $p_n = 1/(n+1)$, $t_n$ reduces to the familiar harmonic mean.

## 2. INTRODUCTION

Let $f(x)$ be a Lebesgue integrable function in the interval $[-1, 1]$. The Legendre series associated with this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n c_n(x) \equiv \Sigma\, V_n(x) \qquad \dots(2.1)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(y)\, c_n(y)\, dy \qquad \dots(2.2)$$

where $c_n(x)$ is the $n$th Legendre polynomial.

Dealing with absolute Nörlund summability factor of Legendre series Beohar (1969) has proved the following:

*Theorem A* — If $f(x)$ is of bounded variation in $[-1, 1]$, then the series

$$\sum_{n=1}^{\infty} \frac{(n+1)\, p_n}{P_n} V_n(x) \qquad \ldots(2.3)$$

is absolutely summable by Nörlund means at a point $x$ of the interval $(-1, 1)$, where the sequence $\{p_n\}$ is real, non-negative and non-increasing such that,

(i) the sequence $\left\{\frac{(n+1)\, p_n}{P_n}\right\}$ is of bounded variation.

(ii) the sequence $\{p_n - p_{n+1}\}$ is non-increasing.

In the present paper we improve the above theorem of Beohar by proving the theorem without using the hypothesis (ii). In what follows we prove the following theorem:

*Theorem* — If $f(x)$ is of bounded variation in $[-1, 1]$, then the series

$$\sum_{n=1}^{\infty} \frac{(n+1)\, p_n}{P_n} V_n(x) \qquad \ldots(2.4)$$

is summable $| N, p_n |$ at an internal point $x$ of the interval $(-1, 1)$, where the sequence $\{p_n\}$ is real non-negative and non-increasing such that the sequence

$$\psi_n = \left\{\frac{(n+1)\, p_n}{P_n}\right\}$$

is of bounded variation.

## 3. Lemmas

We need the following lemmas for the proof of the theorem.

*Lemma* 1 — If $\{p_n\}$ is a non-negative and non-increasing sequence then for $0 < t < 2\pi$ and for every $n$, $a$ and $b$ we have

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| < A P_\tau \qquad \ldots(3.1)$$

where $A$ is an absolute constant $\tau = [1/t]$ and $[x]$ denote the integral part in $x$.

For the proof see McFadden (1942).

*Lemma* 2 — Let $\{p_n\}$ be non-negative and non-increasing therefore for $0 \leqslant \mu \leqslant \nu$, $(\nu \geqslant 1)$

$$\sum_{n=\nu}^{\infty} \frac{p_{n-\mu}}{(n+1) P_n} = O\left(\frac{1}{\nu}\right) \text{ (see Dass 1969).} \qquad \ldots(3.2)$$

*Lemma* 3 — For $0 \leqslant \theta \leqslant \pi$

$$| c_{n+1} (\cos \theta) - c_{n-1} (\cos \theta) | \leqslant m \sqrt{\sin \theta / n} \qquad \ldots(3.3)$$

$m$ is independent of $n$ and $\theta$. (see Obrechkoff 1936).

*Lemma* 4 — For $\epsilon \leqslant \theta \leqslant \pi - \epsilon$, $0 < \epsilon < \frac{1}{2}\pi$; $\epsilon_1 \leqslant \phi \leqslant \pi - \epsilon_1$, $0 < \epsilon_1 < \frac{1}{2}\pi$

$$[c_{n-k+1} (\cos \phi) - c_{n-k-1} (\cos \phi)]\, c_{n-k} (\cos \theta)$$

$$= \frac{2}{\pi(n-k)} \sqrt{\frac{\sin \phi}{\sin \theta}} [\sin (n - k + \tfrac{1}{2}) (\phi - \theta)$$

$$+ \cos (n - k + \tfrac{1}{2}) (\theta + \phi)] + O(n-k)^{-2} \qquad \ldots(3.4)$$

(see Sansone 1959)

$$c_n (\cos \theta) = \sqrt{\frac{2}{\pi n \sin \theta}} \cos \left\{(n + \tfrac{1}{2}) \theta - \frac{\pi}{4}\right\} + O(n^{-3/2}), \epsilon \leqslant \theta \leqslant \pi - \epsilon. \qquad \ldots(3.5)$$

## 4. Proof of the Theorem

We have

$$t_n - t_{n-1} = \sum_{\nu=0}^{n-1} \left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right) V_{n-\nu}(x) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}}$$

$$\times (n - \nu + \tfrac{1}{2})\, c_{n-\nu}(x) \int_{-1}^{1} f(y)\, c_{n-\nu}(y)\, dy$$

$$= \frac{1}{2P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}}$$

$$\times c_{n-\nu}(x) \int_{-1}^{1} f(y) \left\{\frac{d}{dy} c_{n-\nu+1}(y) - \frac{d}{dy} c_{n-\nu-1}(y)\right\} dy.$$

Now in view of the relation

$$c_n(1) = 1, \; c_n(-1) = (-1)^n$$

in order to prove the theorem we have to show that

$$\sum_n \frac{1}{P_n P_{n-1}} \left| \int_{-1}^{1} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}} \, . \, c_{n-\nu}(x) \right.$$

$$\left. \times \{c_{n-\nu+1}(y) - c_{n-\nu-1}(y)\} \, df(y) \right| < \infty. \quad \ldots(4.1)$$

We set $x = \cos\theta$, $y = \cos\phi$ and the above expression becomes

$$\sum_n \frac{1}{P_n P_{n-1}} \left| \int_{0}^{\pi} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}} \, c_{n-\nu}(\cos\theta) \right.$$

$$\left. \times \{c_{n-\nu+1}(\cos\phi) - c_{n-\nu-1}(\cos\phi)\} \, df(\cos\phi) \right| \quad \ldots(4.2)$$

$$= \sum_n \frac{1}{P_n P_{n-1}} \left| \int_0^{c/\sqrt{n}} + \int_{c/\sqrt{n}}^{\theta-(cp_n/P_n^2)} + \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \right.$$

$$\left. + \int_{\theta+(cp_n/P_n^2)}^{\pi-(c/\sqrt{n})} + \int_{\pi-(c/\sqrt{n})}^{\pi} \right|$$

$$\leqslant J_1 + J_2 + J_3 + J_4 + J_5, \text{ say.}$$

We have

$$J_1 \leqslant \sum_n \frac{1}{P_n P_{n-1}} \left| \int_0^{c/\sqrt{n}} \sum_{\nu=0}^{n-1} P_n p_\nu \frac{m\sqrt{\sin\phi}}{(n-\nu)} \, df(\cos\phi) \right|$$

$$\leqslant \sum_n \frac{1}{P_n P_{n-1}} \frac{m}{n^{1/4}} \sum_{\nu=0}^{n-1} \frac{P_n p_\nu}{n-\nu}$$

$$= \sum_n \frac{1}{P_n P_{n-1}} \frac{m}{n^{1/4}} \frac{1}{n+1} \sum_{\nu=0}^{n-1} \left(\frac{1}{n-\nu} + \frac{1}{\nu+1}\right) (\nu+1) \, P_n p_\nu$$

$$= O(1). \quad \ldots(4.3)$$

We now consider

$$J_3 \leqslant \sum_{1}^{\tau} + \sum_{\tau+1}^{\infty} = J_{3\cdot 1} + J_{3\cdot 2}, \text{ where } \tau = \left[\frac{1}{\theta + \phi}\right],$$

$$J_{3\cdot 1} \leqslant \sum_{1}^{\tau} \frac{\sqrt{\sin \phi}}{P_n P_{n-1}} P_n P_{n-1} \sum_{\nu=0}^{n-1} \frac{1}{(\nu + 1)(n - \nu)} = O(1). \qquad \text{...(4.4)}$$

Also

$$J_{3\cdot 2} = \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n - \nu + 1) p_{n-\nu}}{P_{n-\nu}} \right.$$

$$\times \left[\frac{2}{\pi(n - \nu)} \sqrt{\frac{\sin \phi}{\sin \theta}} \{\sin (n - \nu + \tfrac{1}{2})(\phi - \theta)\right.$$

$$\left. \left. + \cos (n - \nu + \tfrac{1}{2})(\theta + \phi)\} + O(n - \nu)^{-2}\right] df(\cos \phi) \right|$$

$$\leqslant J_{3\cdot 2\cdot 1} + J_{3\cdot 2\cdot 2} + J_{3\cdot 2\cdot 3}, \text{ say}$$

Now

$$J_{3\cdot 2\cdot 2} \leqslant J_{3\cdot 2\cdot 2\cdot 1} + J_{3\cdot 2\cdot 2\cdot 2}$$

where

$$J_{3\cdot 2\cdot 2\cdot 1} = \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \sum_{\nu=0}^{[n/2]-1} \left(P_n - \frac{P_\nu p_n}{p_\nu}\right) p_\nu \frac{(n-\nu+1) p_{n-\nu}}{P_{n-\nu}} \right.$$

$$\left. \times \frac{2}{\pi(n - \nu)} \sqrt{\frac{\sin \phi}{\sin \theta}} \{\cos (n - \nu + \tfrac{1}{2})(\theta + \phi)\} \, df(\cos \phi) \right|.$$

Now by Abel's transformation and the hypothesis

$$J_{3\cdot 2\cdot 2\cdot 1} = O\left\{ \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left[ P_\tau \left(P_n - \frac{p_n P_{[n/2]-1}}{p_{[n/2]-1}}\right) \right. \right.$$

$$\left. \left. \times \frac{(n - [n/2] + 2) \, p_{n-[n/2]+1}}{(n - [n/2] + 1) \, P_{n-[n/2]+1}} \right] \right\}$$

$$+ O\left[ \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} P_\tau \sum_{\nu=0}^{[n/2]-2} \left| \Delta \left\{ (P_n - p_n P_\nu / p_\nu) \right. \right. \right.$$

(*equation continued on p. 660*)

$$\times \frac{(n-\nu+1)\,p_{n-\nu}}{(n-\nu)\,P_{n-\nu}}\Big\}\Big|\Big]$$

$$= O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \{P_\tau p_{n-[n/2]+1}\}\Big] + O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}}$$

$$\times \Big\{P_\tau \sum_{\nu=0}^{[n/2]-2} \Big(P_n - \frac{p_n P_\nu}{p_\nu}\Big) \frac{(n-\nu+1)}{(n-\nu)}\Big|\triangle\Big(\frac{p_{n-\nu}}{P_{n-\nu}}\Big)\Big|\Big\}\Big]$$

$$+ O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \Big\{P_\tau p_n \sum_{\nu=0}^{[n/2]-2} \frac{(n-\nu+1)\,p_{n-\nu}}{(n-\nu)\,P_{n-\nu}} \Big(\frac{P_{\nu+1}}{p_{\nu+1}} - \frac{P_\nu}{p_\nu}\Big)\Big\}\Big]$$

$$+ O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \Big\{P_\tau \sum_{\nu=0}^{[n/2]-2} \frac{p_{n-\nu}}{P_{n-\nu}} \Big(P_n - \frac{p_n P_\nu}{p_\nu}\Big)$$

$$\times \frac{1}{(n-\nu)(n-\nu+1)}\Big\}\Big]$$

$$= O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} (P_\tau p_n)\Big] + O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \Big\{P_\tau P_n \frac{p_{n-[n/2]+1}}{P_{n-[n/2]+1}}\Big\}\Big]$$

$$+ O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \Big\{P_\tau p_n \frac{p_{n-[n/2]+2} P_{[n/2]-1}}{P_{n-[n/2]+2}\, p_{[n/2]-1}}\Big\}\Big]$$

$$= O\Big[\sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} P_\tau p_{[n/2]}\Big].$$

Therefore

$$J_{3\cdot2\cdot2\cdot1} = O\Big[P_\tau \sum_{\tau+1}^{\infty} \frac{p_{[n/2]}}{P_n P_{n-1}}\Big] = O(1). \qquad \ldots(4.5)$$

We now consider $J_{3\cdot2\cdot2\cdot2}$. Since

$$\frac{1}{P_n P_{n-1}} (P_n p_\nu - P_\nu p_n) = \frac{1}{(n+1)\,P_{n-1}} \{P_\nu(\psi_\nu - \psi_n) + p_\nu(n-\nu)\}$$

where $\quad \psi_n = \dfrac{(n+1)\,p_n}{P_n}$

$$J_{3\cdot2\cdot2\cdot2} \leqslant \sum_{\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \sum_{\nu=[n/2]}^{n-1} P_\nu(\psi_\nu - \psi_n) \frac{(n-\nu+1)\,p_{n-\nu}}{P_{n-\nu}} \right.$$

$$\left. \times \frac{2}{\pi(n-\nu)} \sqrt{\frac{\sin\phi}{\sin\theta}} \cos\left(n-\nu+\tfrac{1}{2}\right)(\theta+\phi)\, df(\cos\phi) \right|$$

$$+ \frac{2}{\pi} \sum_{\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \sum_{\nu=[n/2]}^{n-1} p_\nu \sqrt{\frac{\sin\phi}{\sin\theta}} \right.$$

$$\left. \times \frac{(n-\nu+1)\,p_{n-\nu}}{P_{n-\nu}} \cos\left(n-\nu+\tfrac{1}{2}\right)(\theta+\phi)\, df(\cos\phi) \right|$$

$$\leqslant J'_{3\cdot2\cdot2\cdot2} + J''_{3\cdot2\cdot2\cdot2}, \text{ say.}$$

Now by Abel's transformation

$$\sum_{\nu=[n/2]}^{n-1} P_\nu(\psi_\nu - \psi_n) \frac{\psi_{n-\nu}}{n-\nu} \cos\left(n-\nu+\tfrac{1}{2}\right)t, \quad \text{where} \quad t = (\theta+\phi)$$

$$= O\left(\frac{1}{t}\right) \sum_{\nu=[n/2]}^{n-2} P_{\nu+1}\, |\Delta\psi_\nu|\, \frac{p_{n-\nu}}{P_{n-\nu}} + O\left(\frac{1}{t}\right)$$

$$\times \sum_{\nu=[n/2]}^{n-2} P_{\nu+1}\, |\psi_{\nu+1} - \psi_n| \left| \Delta\left(\frac{(n-\nu+1)\,p_{n-\nu}}{(n-\nu)\,P_{n-\nu}}\right)\right|$$

$$+ O\left[\frac{P_{n-1}}{t}(\psi_{[n/2]} - \psi_n)\right]$$

$$= O\left(\frac{1}{t}\right) \sum_{\nu=[n/2]}^{n-2} P_{\nu+1}\, |\Delta\psi_\nu| + O\left(\frac{P_{n-1}}{t}\right)$$

$$\times \sum_{\nu=[n/2]}^{n-2} \left|\Delta\left(\frac{p_{n-\nu}}{P_{n-\nu}}\right)\right| \sum_{\mu=\nu}^{n-2} |\Delta\psi_\mu|$$

$$+ O\left(\frac{P_{n-1}}{t}\right) \sum_{\nu=[n/2]}^{n-2} \frac{p_{n-\nu-1}}{(n-\nu)^2\, P_{n-\nu-1}} \sum_{\mu=\nu}^{n-2} |\Delta\psi_\mu|$$

$$+ O\left[\frac{P_{n-1}}{t}(\psi_{[n/2]} - \psi_n)\right]$$

$$= O\left[\frac{P_{n-1}}{t}\sum_{\nu=[n/2]}^{n-2} |\triangle \psi_\nu|\right] + O\left[\frac{P_{n-1}}{t}(\psi_{[n/2]} - \psi_n)\right].$$

Hence

$$J'_{3\cdot2\cdot2\cdot2} = O\left(\frac{1}{t}\right)\left[\sum_{\tau+1}^{\infty}\frac{1}{(n+1)}\sum_{\nu=[n/2]}^{n-2} |\triangle \psi_\nu|\right]$$

$$= O\left(\sin\frac{t}{2}\right)^{-1}\left[\sum_{\tau+1}^{\infty}\frac{1}{(n+1)}\sum_{\nu=[n/2]}^{n-2} |\triangle \psi_\nu|\right]$$

where $O\left(\frac{1}{t}\right) = O(\sin\tfrac{1}{2}t)^{-1} = O(1).$

Thus

$$J'_{3\cdot2\cdot2\cdot2} = O\left[\sum_{\tau+1}^{\infty}\frac{1}{(n+1)}\sum_{\nu=[n/2]}^{n-2} |\triangle \psi_\nu|\right] = O(1) \text{ (Nand Kishor 1967).} \quad \ldots(4.6)$$

Now

$$J''_{3\cdot2\cdot2\cdot2} = \frac{2}{\pi}\sum_{\tau+1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{\nu=[n/2]}^{n-1} p_\nu \frac{(n-\nu+1)\,p_{n-\nu}}{P_{n-\nu}}\right.$$

$$\left.\times \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \cos(n-\nu+\tfrac{1}{2})(\theta+\phi)\,df(\cos\phi)\right|.$$

Again by using Abel's transformation, we get

$$= O\left(\frac{1}{t}\right)\sum_{\tau+1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left\{\sum_{\nu=[n/2]}^{n-2}(p_\nu - p_{\nu+1})\psi_{n-\nu}\right.$$

$$\left.+\sum_{\nu=[n/2]}^{n-2} p_{\nu+1}|\triangle \psi_{n-\nu}|\right\} + O\left(\frac{1}{t}\right)\sum_{\tau+1}^{\infty}\frac{p_{n-1}}{(n+1)P_{n-1}}$$

$$\leqslant O\left(\frac{1}{t}\right)\left[\sum_{\tau+1}^{\infty}\frac{p_{[n/2]}}{(n+1)P_{n-1}}\right] + O\left(\frac{1}{t}\right)\left[\sum_{\tau+1}^{\infty}\frac{p_{n-1}}{(n+1)P_{n-1}}\right]$$

$$= O(1). \quad \ldots(4.7)$$

Thus we see that

$$J_{3\cdot 2\cdot 2} = O(1). \qquad \text{...(4.8)}$$

Also

$$J_{3\cdot 2\cdot 1} = \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta-(cp_n/P_n^2)}^{\theta+(cp_n/P_n^2)} \sum_{\nu=0}^{n-1} P_n p_\nu \mid \phi - \theta \mid \; \mid df(\cos\phi) \mid \right|$$

$$= \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} P_n p_\nu (cp_n/P_n^2) = O(1) \qquad \text{...(4.9)}$$

and obviously we get

$$J_{3\cdot 2\cdot 3} = O(1) \qquad \text{...(4.10)}$$

and thus

$$J_3 = O(1). \qquad \text{...(4.11)}$$

We shall now consider

$$J_2 = \sum_{1}^{e} + \sum_{e+1}^{\infty} \leqslant J_{2\cdot 1} + J_{2\cdot 2}, \text{ say}$$

where $e$ is any positive integer.

Now

$$J_{2\cdot 1} = O(1) \qquad \text{...(4.12)}$$

as in case of $J_{3\cdot 1}$.

Now to calculate $J_{2\cdot 2}$ we use the following asymptotic expression of $c_n(\cos\phi)$, for $c/\sqrt{n} \leqslant \phi \leqslant \pi - c/\sqrt{n}$

$$c_n(\cos\phi) = \sqrt{\frac{2}{4\pi \sin\phi}} \cos\left\{(n + \tfrac{1}{2})\phi - \frac{\pi}{4}\right\} + (n \sin\phi)^{-1} O(1),$$

and therefore we get

$$c_{n-\nu}(\cos\theta)\{c_{n-\nu+1}(\cos\phi) - c_{n-\nu-1}(\cos\phi)\}$$

$$= \frac{2}{\pi(n-\nu)} \sqrt{\frac{\sin\phi}{\sin\theta}} \{\cos(n - \nu + \tfrac{1}{2})(\theta + \phi)$$

$$+ \sin(n - \nu + \tfrac{1}{2})(\phi - \theta)\} + O(n-\nu)^{-2} \sin^{-3/2}\phi.$$

Thus

$$J_{2\cdot 2} = \sum_{e+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{c/\sqrt{n}}^{\theta-(cp_n/P_n^2)} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \frac{(n-\nu+1)\, p_{n-\nu}}{P_{n-\nu}} \right.$$

$$\times \left[ \frac{2}{\pi(n-\nu)} \sqrt{\frac{\sin\phi}{\sin\theta}} \{\cos(n-\nu+\tfrac{1}{2})(\theta+\phi)\right.$$

$$\left. \left. + \sin(n-\nu+\tfrac{1}{2})(\phi-\theta)\} + O(n-\nu)^{-2}\sin^{-3/2}\phi \right] df(\cos\phi) \right|$$

$$= \sum_{e+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{c/\sqrt{n}}^{\theta-(cp_n/P_n^2)} \left\{ \sum_{\nu=0}^{[n/2]-1} + \sum_{[n/2]}^{n-1} \right\} \cdots \right|$$

$$\leqslant J_{2\cdot 2\cdot 1} + J_{2\cdot 2\cdot 2}.$$

Also

$$J_{2\cdot 2\cdot 1} \leqslant J_{2\cdot 2\cdot 1\cdot 1} + J_{2\cdot 2\cdot 1\cdot 2} + J_{2\cdot 2\cdot 1\cdot 3}.$$

Proceeding as in $J_{3\cdot 2\cdot 2\cdot 1}$ we can show that

$$J_{2\cdot 2\cdot 1\cdot 1} = O(1). \qquad \text{...(4.13)}$$

Now to calculate $J_{2\cdot 2\cdot 1\cdot 2}$ we choose $e = \left[\frac{1}{\theta-\phi}\right]$ and following the procedure adopted in $J_{3\cdot 2\cdot 2\cdot 1}$ we get

$$J_{2\cdot 2\cdot 1\cdot 2} = O(1) \qquad \text{...(4.14)}$$

and easily

$$J_{2\cdot 2\cdot 1\cdot 3} = O(1). \qquad \text{...(4.15)}$$

Thus we see that

$$J_{2\cdot 2\cdot 1} = O(1). \qquad \text{...(4.16)}$$

$J_{2\cdot 2\cdot 2}$ can also be treated in a similar manner. $J_4$ and $J_5$ can be disposed of in exactly the same way as $J_2$ and $J_1$, respectively.

This completes the proof of the theorem.

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