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A THEOREM ON THE ZEROS OF CERTAIN CLASSICAL POLYNOMIALS

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Recently Wragg and Underhill (1976) have derived a matrix of order $n \times n$, the eigenvalues of which are zeros of the Bessel polynomials. In the present paper, a general result of this kind is proved by making use of the Crammer's rule. In addition to the Bessel polynomials our theorem may also be applied to the polynomials of Legendre, Hermite, Laguerre and Jacobi.

1. THE MAIN RESULT

We prove the following theorem:

Theorem — Let $\{f_n(x)\}$ be a simple set of polynomials which obey the following pure recurrence relation:

$$a_n f_n(x) + (b_n - x) f_{n-1}(x) + c_n f_{n-2}(x) = 0 \qquad ...(1)$$

for $n \geqslant 2$ with $f_0(x) = 1$, $af_1(x) = b + x$, where a_n, b_n and c_n are expressions in n which are independent of x, or otherwise constants, with $a_n \neq 0$, $c_n \neq 0$; a and b are arbitrary constants, independent of x and n both with $a \neq 0$. Then

$$f_n(x) = \frac{(-1)^n}{a\, a_2 a_3 \ldots a_n} \det A \qquad ...(2)$$

where the matrix $A_{n\times n} = \{\alpha_{ij}\}_{n\times n}$ is given by

$$\alpha_{ij} = \begin{cases} 0 & \text{when } j \notin \{i-1, i, i+1\}, & 1 \leqslant i \leqslant n \\ a_{n-i+1} & j = i - 1, & 1 \leqslant i \leqslant n-1 \\ b_{n-i+1-x} & j = i, & 1 \leqslant i \leqslant n-1 \\ c_{n-i+1} & j = i+1, & 1 \leqslant i \leqslant n-1 \\ a & j = i-1, & i = n \\ -(b+x) & j = i, & i = n. \end{cases}$$

...(3)

Moreover, the zeros of $f_n(x)$ are the eigenvalues of the matrix $B_{n\times n} = \{\beta_{ij}\}_{n\times n}$ defined by

$$\beta_{ij} = \begin{cases} \alpha_{ij} & \text{if } i \neq j, \\ b_{n-i+1} & \text{if } i = j \text{ and } 1 \leqslant i \leqslant n-1, \\ -b & \text{if } i = j \text{ and } i = n. \end{cases} \qquad \ldots(4)$$

2. Proof

Making use of the recurrence relation (1) together with the expressions for $f_0(x)$ and $f_1(x)$, we form a system of n linear equations for n unknowns $f_1(x), f_2(x), \ldots, f_n(x)$:

$$\left.\begin{aligned} &a_n f_n(x) + (b_n - x)\, f_{n-1}(x) + c_n f_{n-2}(x) = 0 \\ &a_{n-1} f_{n-1}(x) + (b_{n-1} - x)\, f_{n-2}(x) + c_{n-1} f_{n-3}(x) = 0 \\ &a_{n-2} f_{n-2}(x) + (b_{n-2} - x)\, f_{n-3}(x) + c_{n-2} f_{n-4}(x) = 0 \\ &\ldots \\ &a_2 f_2(x) + (b_2 - x)\, f_1(x) + c_2 = 0 \\ &\qquad a f_1(x) = b + x. \end{aligned}\right\} \qquad \ldots(5)$$

Cramer's rule when applied to the eqns. (5) yields

$$f_n(x) = \frac{(-1)^n}{a\, a_2 a_3 \ldots a_n}$$

$$\det \begin{bmatrix} (b_n - x) & c_n & 0 & 0 & \ldots & 0 & 0 & 0 \\ a_{n-1} & (b_{n-1} - x) & c_{n-1} & 0 & \ldots & 0 & 0 & 0 \\ 0 & a_{n-1} & (b_{n-2} - x) & c_{n-2} & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & a_2 & (b_2 - x) & c_2 \\ 0 & 0 & 0 & 0 & \ldots & 0 & a & -(b + x) \end{bmatrix} \qquad \ldots(6)$$

and the theorem follows immediately.

3. Concluding Remarks

The theorem proved above would readily yield the corresponding results for the polynomials of Legendre, Hermite, Laguerre and Jacobi by considering their recurrence relations [see for instance Rainville {1960, (2), p. 160; (7), p. 188; (4), p. 202; (1), p. 263 respectively}]. Using Barnes (1973) formula of the type (1) for the Bessel polynomials, we obtain the result of Wragg and Underhill (1976). It is also interesting to remark here that for $a_n = -1$, $b_n = -\{(n-1)\,a + b\}$, $c_n = -(n-1)\,ab$, $a = -1$, our theorem would reduce to the main result of Frost and Sackfield (1975) which they have obtained by employing a variation on the usual generating function technique.

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