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EFFICIENCY IN CERTAIN NONLINEAR FRACTIONAL VECTOR MAXIMIZATION PROBLEMS

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In this note a necessary condition for a point to be efficient is derived for a certain class of nonlinear fractional vector maximization problems.

INTRODUCTION

In this note we consider a certain special class of vector maximization problems, namely, nonlinear fractional vector maximization problems. The concept of efficiency has been defined for this class of problems and a necessary condition for a point to be efficient is derived for the same. This extends the results of Kaul and Gupta (1979) establishing similar results for the class of linear fractional vector maximization problems.

NONLINEAR FRACTIONAL VECTOR MAXIMUM PROBLEM

Consider the nonlinear fractional vector valued criterion function

$$h(x) = \left[\frac{f^1(x)}{g^1(x)}, \frac{f^2(x)}{g^2(x)}, \ldots, \frac{f^k(x)}{g^k(x)}\right] \qquad \ldots(1)$$

defined over the set X given by

$$X = \{x \in R^n : Ax \leqslant b, x \geqslant 0\} \qquad \ldots(2)$$

where A is an $m \times n$ matrix and b is a constant m-vector. Let us denote by K the index set $\{1, 2, \ldots, k\}$. Each $f^i(x)$, $i \in K$ is a convex differentiable function of x and each $g^i(x)$ for $i \in K$ is a concave differentiable function of x. It is assumed that $g^i(x)$ is positive over X for all $i \in K$. It is further assumed that $f^i(x) \geqq 0$ for all $i \in K$ and all $x \in X$. Throughout this note $\nabla f^i(x)$ denotes the $n \times 1$ gradient vector and

$\nabla f(\dot{x}) = [\nabla f^1(x), \nabla f^2(x), \ldots, \nabla f^k(x)]$, the $n \times k$ matrix. The nonlinear fractional vector maximum problem

(NFVMP) Maximize $h(x)$

subject to $x \in X$

is a problem of finding all points that are efficient (Geoffrion 1968). A point x^0 is said to be efficient if $x^0 \in X$ and there exists no other point $x^* \in X$, such that $h(x^*) \geqslant h(x^0)$ and $h(x^*) \neq h(x^0)$. Now we will obtain the necessary condition for a point to be efficient for the (NFVMP). The corresponding condition for the linear vector maximum problem, as studied by Isermann (1974) are both necessary and sufficient.

We first establish the following lemma. The vectors $\beta = (\beta_1, \beta_2, \ldots, \beta_k)^t$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)^t$ being used in the lemma have all positive components.

Lemma — Let $x^0 \in X$ be an efficient solution of the NFVMP, then it is necessary that the linear program :

$$\text{(LP — 1)} \quad \text{Maximize } \sum_{i=1}^{k} \beta_i y_i + \sum_{i=1}^{k} \gamma_i z_i$$

$$\text{subject to } Ax \leqslant b$$

$$x^t \nabla f^i(x^0) - y_i = x^{0t} \nabla f^i(x^0), \quad i \in K$$

$$x^t \nabla g^i(x^0) + z_i = x^{0t} \nabla g^i(x^0), \quad i \in K$$

$$x \geqslant 0$$

$$y \geqslant 0$$

$$z \geqslant 0$$

has an optimal solution x^*, y^*, z^* with $y^* = z^* = 0$.

PROOF : Let us assume to the contrary that x^*, y^*, z^* is an optimal solution to (LP — 1) with $y_i^* \neq 0$ for some $i = k$ (say). Then the feasibility of x^*, y^*, z^* implies that

$$x^{*t} \nabla f^k(x^0) > x^{0t} \nabla f^k(x^0) \qquad \ldots(3)$$

$$x^{*t} \nabla f^i(x^0) \geqslant x^{0t} \nabla f^i(x^0), \quad i \in K, \quad i \neq k, \qquad \ldots(4)$$

and

$$x^{*t} \nabla g^i(x^0) \leqslant x^{0t} \nabla g^i(x^0), \quad i \in K. \qquad \ldots(5)$$

Now since each $f^i(x)$ for $i \in K$ is convex, therefore, in particular, the convexity of $f^k(x)$ implies that

$$f^k(x^*) - f^k(x^0) \geqslant (x^* - x^0)^t \nabla f^k(x^0)$$

$$> 0$$

where the last inequality follows on using (3).

Hence

$$f^k(x^*) > f^k(x^0). \qquad \ldots(6)$$

Also each $f^i(x)$, $i \in K$ is convex, we have $\forall\ i \in K$, $i \neq k$,

$$f^i(x^*) - f^i(x^0) \geqslant (x^* - x^0)^t \nabla f^i(x^0) \geqslant 0 \qquad \text{[on using (4)].}$$

Thus

$$f^i(x^*) \geqslant f^i(x^0), \quad i \in K, \quad i \neq k. \qquad \ldots(7)$$

Again concavity of the functions $g^i(x)$, $\forall\ i \in K$, and the relation (5) yield the inequality

$$g^i(x^*) - g^i(x^0) \leqslant (x^* - x^0)^t \nabla g^i(x^0) \leqslant 0.$$

Thus

$$g^i(x^*) \leqslant g^i(x^0)\ \forall\ i \in K. \qquad \ldots(8)$$

Hence using (6), (7), (8) and the assumption that $f^i(x) \geqq 0\ \forall\ i \in K$ and $\forall\ x \in X$, we have

$$h(x^*) \geqslant h(x^0) \quad \text{and} \quad h(x^*) \neq h(x^0),$$

contradicting the fact that x^0 is efficient to the (NFVMP). The condition $z_j^* \neq 0$ for some j, similarly leads to a contradiction. This completes the proof of the lemma.

We now come to establish the necessary condition for a point to be efficient for the (NFVMP), in the following theorem.

Theorem — If x^0 is an efficient solution of the (NFVMP), then there exist vectors u^*, v^* with strictly positive components, such that

$$u^{*t} \nabla^t f(x^0)\, x^0 - v^{*t} \nabla^t g(x^0)\, x^0 \geqq u^{*t} \nabla^t f(x^0)\, x - v^{*t} \nabla^t g(x^0)\, x \quad \forall\ x \in X$$

PROOF : The dual of (LP — 1) is

$$\text{Minimize } w^t b - u^t [\nabla^t f(x^0)]\, x^0 + v^t [\nabla^t g(x^0)]\, x^0$$

$$\text{(DP — 1)} \qquad \text{subject to } w^t A - u^t \nabla^t f(x^0) + v^t \nabla^t g(x^0) \geqslant 0 \qquad \ldots(9)$$

$$u \geqslant \beta > 0$$

$$v \geqslant \gamma > 0$$

$$w \geqslant 0.$$

According to the lemma proved, there exists an optimal solution (x^*, y^*, z^*) to (LP — 1) such that $y^* = z^* = 0$. By the duality theory, there exists a vector (u^*, v^*, w^*) which optimizes the dual problem (DP — 1) such that

$$w^{*t}b - u^{*t}[\nabla^t f(x^0)]\, x^0 + v^{*t}[\nabla^t g(x^0)]\, x^0 = \sum_{i=1}^{k} \beta_i y_i^* + \sum_{i=1}^{k} \gamma_i z_i^* = 0. \quad \ldots(10)$$

It is easy to show that w^* is an optimal solution of the problem.

$$\text{(LP — 2)} \quad \left.\begin{array}{l} \text{Minimize } w^t b \\ \text{subject to } w^t A \geqslant u^{*t}\nabla^t f(x^0) - v^{*t}\nabla^t g(x^0) \\ \qquad w \geqslant 0. \end{array}\right\} \quad \ldots(11)$$

The dual to (LP — 2) is the problem

$$\text{(DP — 2)} \quad \left.\begin{array}{l} \text{Maximize } u^{*t}[\nabla^t f(x^0)]\, y - v^{*t}[\nabla^t g(x^0)]\, y \\ \text{subject to } Ay \leqslant b \\ \qquad y \geqslant 0. \end{array}\right\} \quad \ldots(12)$$

A consequence of the condition (10) above, i.e.

$$w^{*t}b = u^{*t}[\nabla^t f(x^0)]\, x^0 - v^{*t}[\nabla^t g(x^0)]\, x^0$$

is that the vector x^0 is indeed optimal for (DP — 2).

Thus $$u^{*t}[\nabla^t f(x^0)]\, x^0 - v^{*t}[\nabla^t g(x^0)]\, x^0 \geqslant u^{*t}[\nabla^t f(x^0)]\, x - v^{*t}[\nabla^t g(x^0)]\, x \quad \forall\, x \in X,$$

which we were required to obtain.

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