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EQUATORIAL PROPAGATION OF AXISYMMETRIC MAGNETO-RADIATIVE SHOCKS

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In the present paper an exact self-similar solution for the equatorial propagation of axisymmetric, spherical piston-driven and thermally radiated magneto-gasdynamic shocks into an inhomogeneous ideal gas permeated by a current free azimuthal magnetic field is considered. The ambient gas ahead of the shock wave is supposed cool and the shock wave is transparent while the piston is cold. Considerable changes in the flow patterns behind the shock are observed due to the thermal radiation effects.

1. INTRODUCTION

The thermal radiation effects become important at a very high temperature for problems in fluid dynamics. Wang (1964) has discussed the 'piston problem' with radiation energy transfer in the thick and thin limits and also in the general case with the idealised two direction approximation. Elliot (1960) has considered similarity methods in radiation hydrodyamics, particularly in the problem of an intense explosion. Rosenau and Frankenthal (1976) have also given the basic geometry of an axisymmetric piston driven disturbance in the equatorial plane and a double shock structure. He has considered the self-similar solutions for the equatorial propagation of axisymmetric, piston driven magnetohydrodynamic shocks into an inhomogeneous ideal gas permeated by a current free azimuthal magnetic field and implications regarding the propagation of disturbances in stellar atmosphere.

In the present paper, we have extended the problem treated by Rosenau and Frankenthal to take into account the effects of radiation, namely radiation flux, energy and pressure. Also, the mathematical technique employed by us is different from that of Rosenau and Frankenthal (1976). The type of problem in which we are interested is that of a disturbance bounded by a strong shock advancing into an ambient material. We consider the problem of spherical disturbance bounded by a strong shock wave, which is produced on account of an instantaneous release of a finite amount of energy by an expanding piston. The piston in this case represents any means of propelling the plasma radially outward.

Pai (1958) has found that in the cylindrical case a similar solution exists only if the initial tangential magnetic field (h_0) is proportional to r^{-1}. Summers (1975) has constructed a spherically symmetric model incorporating a self-consistent magnetic field to describe a blast wave in the solar wind caused by the explosive energy release of a solar flare. The shock is assumed to advance into a conducting gas of spatially decreasing density and pervaded by an idealized, spatially decreasing magnetic field. The magnetic field is assumed as an idealized field such that the lines of force lie on hemisphere whose centre is the explosion. Rosenau and Frankenthal (1976) have taken a model which may be regarded as a first approximation to the distant flow of a stellar wind in a narrow conical slab centered on the ecliptic plane. There, the stretching of the dipolar field of a rotating star produces the appropriate field and flow. The radial component of the field behaves as r^{-2} and may be ignored in comparison with the azimuthal field which behaves as r^{-1}, where the flow is primarily radial. The choice of an azimuthal field allows only the possibility of fast shocks. All the above authors have also taken power law density distribution in their papers.

We have assumed that the density varies as r^{-w} where $0 \leqslant w < 3$ and the azimuthal magnetic field as r^{-1}. The form of the initial density distribution is more general than that considered by Lee and Chen (1968). The assumption of a cold ambient gas is equivalent to the statement that the shocks we shall consider are acoustically strong. The assumption that the ambient medium is stationary is restrictive as compared with Lee and Chen (1968), but is expected to have little effect on the flow near the piston which is of prime concern to us. The magnetic field distribution corresponds to a current-free ambient medium. The flow is primarily radial and viscosity is neglected.

2. Equations of Motion in Radiation Magnetogasdynamics

The equations of continuity, motion, magnetic field and energy for spherical symmetry are

$$r^2 \frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial r}(r^2 \sigma u) = 0 \qquad \text{...(2.1)}$$

$$r^2 \sigma \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) + r^2 \frac{\partial p}{\partial r} + \tfrac{1}{2} \frac{\partial}{\partial r}(r^2 h^2) = 0 \qquad \text{...(2.2)}$$

$$\frac{\partial}{\partial t}(rh) + \frac{\partial}{\partial r}(ruh) = 0 \qquad \text{...(2.3)}$$

$$\frac{\partial E}{\partial t} + u \frac{\partial E}{\partial r} + p \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) \left(\frac{1}{\sigma} \right) + \frac{1}{\sigma r^2} \frac{\partial}{\partial r}(F r^2) = 0 \qquad \text{...(2.4)}$$

where σ, u, p, h denote mass density, radial velocity, total pressure and magnetic field transverse to the flow respectively. The total pressure p is the combination of the material pressure p_m and the radiative pressure p_R; F is the heat flux and

$$E = E_m + E_R \qquad \ldots(2.5)$$

where

$$E_m = \frac{p_m}{\sigma(\gamma - 1)}, \quad E_R = \frac{3p_R}{\sigma}, \quad p_m = \sigma RT \qquad \ldots(2.6)$$

and γ and R have their usual meanings.

The radiation flux F is given by

$$F = -\frac{C}{\epsilon\sigma}\frac{\partial p_R}{\partial r} \qquad \ldots(2.7)$$

C being the velocity of light and ϵ the coefficient of opacity. Also,

$$p_m = zp, \quad p_R = (1 - z)\, p$$

where $0 < z < 1$, so that

$$E = \frac{p}{\sigma(K - 1)} \qquad \ldots(2.8)$$

where K is called the Klimshin's coefficient and is given by

$$K = \frac{4(\gamma - 1) + Z(4 - 3\gamma)}{3(\gamma - 1) + Z(4 - 3\gamma)}. \qquad \ldots(2.9)$$

The density distribution in stationary medium is taken as

$$\sigma = \sigma_0 r^{-w} \qquad \ldots(2.10)$$

where σ_0 = constant and $0 \leqslant w < 3$.

The magnetic field distribution h is given by

$$h = h_0 r^{-1} \qquad \ldots(2.11)$$

where h_0 is a constant. The initial conditions are

$$u = p = T = F = 0. \qquad \ldots(2.12)$$

For self-similar motion, the speed v of the shock wave is given by

$$v = Vt^{N-1} \qquad \ldots(2.13)$$

where V is a constant depending on the speed of the piston that obeys a power law time dependence, i.e. $r_p \alpha t^N$.

Then, the Rankine-Hugoniot relations immediately downstream of the shock are,

$$\sigma(v - u) = \sigma_0 r_s^{-w}\, v \qquad \ldots(2.14)$$

$$\sigma(v-u)^2 + p + \tfrac{1}{2}h^2 = \sigma_0 r_s^{-w} v^2 + \tfrac{1}{2}h_0^2 r_s \qquad ...(2.15)$$

$$\tfrac{1}{2}(v-u)^2 + \frac{K}{K-1}\frac{p}{\sigma} + \frac{h^2}{\sigma} = \tfrac{1}{2}v^2 + (h_0^2 r_s^{w-2}/\sigma_0) \qquad ...(2.16)$$

$$(v-u)\,h = v h_0 r_s^{-1} \cdot \qquad ...(2.17)$$

The quantities σ, u, p refer to the state just behind the shock, and $r_s = N^{-1}Vt^N$ denotes the position of shock. It can be easily verified that the eqns. (3.1) to (3.6) given in section 3 below when subjected to the initial conditions (2.10) to (2.12) and the boundary conditions imposed by the piston admit a self-similar solution only if the piston index N and the density index are restricted by the condition

$$N = \frac{2}{4-w}\cdot \qquad ...(2.18)$$

3. Reduction to a Similarity Problem

Equations (2.1) to (2.4) can be reduced to four ordinary differential equations provided that the variables are of the form

$$\sigma(r, t) = \sigma_0 r^{-w}\eta^{w-2}\bar{\sigma}(\eta) \qquad ...(3.1)$$

$$u(r, t) = Vt^{N-1}\bar{u}(\eta) \qquad ...(3.2)$$

$$p(r, t) = \sigma_0 r^{-w}\eta^{w-2}V^2 t^{2(N-1)}\bar{p}(\eta) \qquad ...(3.3)$$

$$h(r, t) = b_0 r^{-1}\bar{h}(\eta) \qquad ...(3.4)$$

$$E(r, t) = V^2 t^{2(N-1)}\,\bar{E}(\eta) \qquad ...(3.5)$$

$$F(r, t) = \sigma_0 r^{-w}\eta^{w-2}V^3 t^{3(N-1)}\bar{F}(\eta) \qquad ...(3.6)$$

where

$$\eta = N(Vt^N)^{-1}\,r \qquad ...(3.7)$$

$$b_0 = (\sigma_0 V_0^2)^{1/2},\ V_0 = V^{1/N}N^{\alpha},\ \alpha = (N-1)/N.$$

Equations (2.1) to (2.4) then become

$$-\,w + (\bar{u}-\eta)\frac{\bar{\sigma}'}{\bar{\sigma}} = -\,(2+\bar{u}') \qquad ...(3.8)$$

$$\bar{\sigma}\,[\alpha\bar{u} + (\bar{u}-\eta)\,\bar{u}'] = \frac{2\bar{p}}{\eta} - \bar{p}' - \bar{h}\bar{h}' \qquad ...(3.9)$$

$$\bar{u}'\bar{h} + (\bar{u}-\eta)\,\bar{h}' = 0 \qquad ...(3.10)$$

$$2\alpha\bar{E} + (\bar{u}-\eta)\,\bar{E}' + \frac{\bar{p}}{\bar{\sigma}}\left(\frac{2\bar{u}}{\eta} + \bar{u}'\right) + \frac{1}{\bar{\sigma}}\,\bar{F}' = 0 \qquad ...(3.11)$$

With the shock speed parameter V treated as a known quantity, eqns. (3.8) – (3.11) may be integrated from the shock location, where $\eta = 1$ as given by eqn. (3.7).

Using the relations (2.18) to (3.4), the Rankine-Hugoniot relations, take the form

$$\bar{\sigma}(\bar{u} - \eta) = -1 \qquad ...(3.12)$$

$$\bar{\sigma}(\bar{u} - \eta)^2 + \tfrac{1}{2}\bar{h}^2 + \bar{p} = 1 + \tfrac{1}{2} M_A^{-2} \qquad ...(3.13)$$

$$(\bar{u} - \eta)\bar{h} = -M_A^{-1} \qquad ...(3.14)$$

$$\tfrac{1}{2}(\bar{u} - \eta)^2 - (\bar{u} - \eta)\bar{h}^2 + \left(\frac{K}{K-1}\right)\frac{\bar{p}}{\bar{\sigma}} = \tfrac{1}{2} + M_A^{-2} \qquad ...(3.15)$$

where

$$M_A^{-1} = h_0/b_0. \qquad ...(3.16)$$

Here M_A behaves as an Alfvén-Mach number which measures the shock speed in terms of the Alfvén velocity defined by the upstream magnetic field at the shock location.

The Alfvén-Mach number M_A is the only physical parameter, besides K and ambient density index w, which affects the problem. M_A may range over the interval, $0 \leqslant M_A^{-2} \leqslant 1$.

At the extremity $M_A^{-2} = 1$, the shock degenerates into a weak discontinuity which does not compress the gas. The extremity $M_A^{-2} = 0$, which corresponds to a vanishing magnetic field, represents the hydrodynamic limit. For magnetically strong shocks, i.e. $M_A^{-2} \to 0$, the solutions of eqns. (3.12) – (3.15) yield the following familiar expressions for an acoustically strong shock, accompanied by a down stream magnetic field $\bar{h}_{(1)}$ (retaining the lowest order term M_A^{-1} only).

$$\bar{\sigma}(1) = (K + 1)/(K - 1) \qquad ...(3.17)$$

$$\bar{p}(1) = 2/(K - 1) \qquad ...(3.18)$$

$$\bar{u}(1) = 2/(K + 1) \qquad ...(3.19)$$

$$\bar{h}(1) = M_A^{-1}(K + 1)/(K - 1). \qquad ...(3.20)$$

We assume the product solution of the 'progressive wave' in the form (Verma and Vishwakrama 1978)

$$u = art^{-1} \quad ...(3.21)$$

$$\sigma = (\lambda + 1)\, f t^{-2\beta} x^{\lambda - 2} \quad ...(3.22)$$

$$p = \beta^2 f t^{-2} b(t)\, x^{\lambda} \quad ...(3.23)$$

$$h = \beta f^{1/2} t^{-1} c(t)\, x^{\lambda/2} \quad ...(3.24)$$

where

$$x = rt^{-\beta} \quad ...(3.25)$$

a is some function of t and $f(t)$, $b(t)$ and $c(t)$ are given by

$$a(t) = \frac{\beta\lambda - tf_t/f}{(\lambda + 1)} = \frac{2 - 2tc_t/c}{3} \quad ...(3.26)$$

$$b(t) + (\tfrac{1}{2} + \lambda^{-1})\, c^2(t) = \frac{\lambda + 1}{\lambda\beta^2}\,[-a^2 + a - ta_t] \quad ...(3.27)$$

β and λ being some constants. It can be easily seen that these equations satisfy (2.1) - (2.3) identically.

After changing this solution to similarity form which requires a to be constant, we apply the above boundary conditions, i.e. eqns. (3.17) to (3.20) and obtain

$$\bar{u}(\eta) = 2\eta(K + 1)^{-1} \quad ...(3.28)$$

$$\bar{\sigma}(\eta) = \eta^{\lambda}(K + 1)\,(K - 1)^{-1} \quad ...(3.29)$$

$$\bar{p}(\eta) = \eta^{\lambda+2} 2K(K + 1)^{-1} \quad ...(3.30)$$

$$\bar{h}(\eta) = (K + 1)\, M_A^{-1}\, \eta^{3\lambda/2}(K - 1)^{-1}. \quad ...(3.31)$$

Using (3.28) and (3.29) in eqn. (3.8) we obtain

$$\lambda = \frac{w(K + 1) - 2(K + 2)}{1 - K}. \quad ...(3.32)$$

From eqn. (3.11) we have

$$\bar{F}'(\eta) = \left[\frac{2wK(K + 1) + 4K(K - 3)}{(K + 1)^2\,(1 - K)}\right] \eta^{\lambda+2} \quad ...(3.33)$$

which after integration from η_p to 1 is given by

$$\bar{F}(\eta) = \frac{2wK(K + 1) + 4K(K - 3)}{(K + 1)^2\,\{w(K + 1) - (5K + 1)\}}\,(\eta^{\lambda+3} - \eta_p^{\lambda+3}) \quad ...(3.34)$$

where the integration constant is determined by assuming the piston to be cold i.e. at $\eta = \eta_p$, the flux $\bar{F} = 0$. As a consequence of the relations (3.29) and (3.30) we have

$$\bar{E}(\eta) = 2\eta^2(K+1)^{-1}. \qquad ...(3.35)$$

The temperature is obtained from

$$T = R^{-1}V^2t^{2(N-1)}\bar{T}(\eta)$$

so that we have,

$$\bar{T}(\eta) = 2\eta^2 K(K-1)(K+1)^{-2}. \qquad ...(3.36)$$

Equations (3.28) - (3.36) give the solution of our problem. Starting at the shock location $\eta = 1$ the integration of eqn. (3.8) - (3.11) proceeds towards the piston (with decreasing value of η) and is terminated at the point $\eta = \eta_p$ where the condition $\bar{u} = \eta$ is satisfied. This condition follows from the definition of the piston as the point where the gas moves at the speed imposed by the piston. This is the singular point.

4. Results and Discussion

While in the non-radiative case two integrals (the frozen-field and entropy integral) are possible, it may be noted that in our case only the frozen-field integral is possible for the system and is given by

$$\bar{h} \mid y \mid^{(2-w)/(3-w)} \bar{\sigma}^{-1/(3-w)} = b_1 \qquad ...(4.1)$$

where the constant b_1 may be obtained from eqns. (3.8) and (3.10) taking y in place of $\bar{u} - \eta$.

If q be the total amount of heat energy liberated per unit mass per unit time over the whole frequency interval,

$$\text{div } F = \sigma q \qquad ...(4.2)$$

which for a spherically symmetric distribution of matter takes the form

$$\sigma q = \frac{1}{r^2}\frac{d}{dr}(r^2F) \qquad ...(4.3)$$

F being the integrated flux across elements of surface normal to the direction of the radius vector r.

The similarity transformation for q is given by

$$q(r, t) = V^2t^{2n-3}\bar{q}(\eta). \qquad ...(4.4)$$

Then $\bar{q}$ is obtained as

$$\bar{q} = \left[\frac{4Kw(K+1) + 8K(K-3)}{(K+1)^3(w-4)}\right]\eta^2. \qquad ...(4.5)$$

From the above expression we see that $\bar{q}$ is a negative quantity. Hence it represents the loss of amount of heat energy liberated by unit mass in unit time.

Now we calculate $\bar{u}$, $\bar{\sigma}$, $\bar{p}$, $\bar{E}$, $\bar{T}$, $\bar{F}$ and $\bar{q}$ taking $z = 0.6$, $\gamma = 1.4$, $w = 2.5$ and $M_A = 10$. Consequently the values of K and λ are obtained as 1.3703703 and 2.2 respectively.

It may be noted that in evaluating $\bar{F}$ we would need η_p which has been obtained as 0.01 after having calculated the values of $\bar{u}$, $\bar{\sigma}$, $\bar{h}$ and $\bar{p}$. From Figs. 1–4, we see that at the piston point, $\bar{p}$, $\bar{\sigma}$, $\bar{h}$, $\bar{E}$, $\bar{T}$, $\bar{F}$, and $\bar{q}$ become almost zero. As η decreases all other quantities decrease continuously. At $\eta = 0.1$, $\bar{p}$, $\bar{h}$ become negligible while at $\eta = 0.04$, $\bar{T}$ and $\bar{E}$ also become negligible. Near $\eta = 0.1$ the flux and the heat energy liberation per unit mass, per unit time at all frequency interval become almost negligible. Near the piston, heat energy liberation does not take place and also there is no magnetic effect there.

The singular point, i.e. $\bar{u} = \eta$ or $\eta = \eta_p$ is near the point $\eta = 0.01$.

Taking different values of γ(2, 5/3, 3/2, 4/3), M_A(10, 100, 1000), w(0.1, 0.2, ..., 2.9 etc.), z(0.1 to 0.9) and K we can see the effects of magnetic field, radiation flux, energy, density, pressure, temperature and liberated heat energy behind the shock and near the piston. At the stage when the material pressure and energy is dominant,

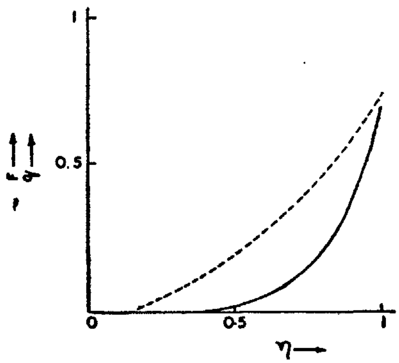

FIG. 1. Variation of flux and heat energy per unit mass per unit time (—— Radiation flux; – – – Heat energy liberation per unit mass per unit time).

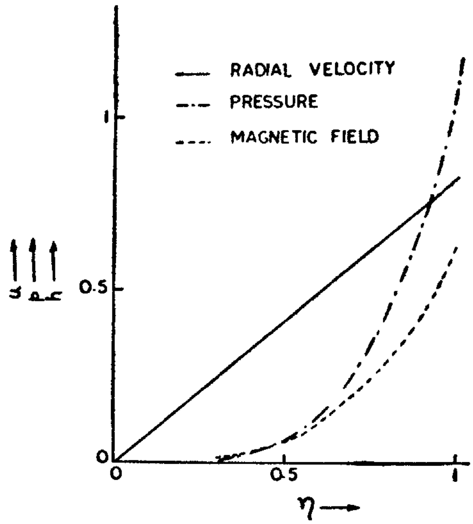

FIG. 2. Variation of velocity, pressure and magnetic field.

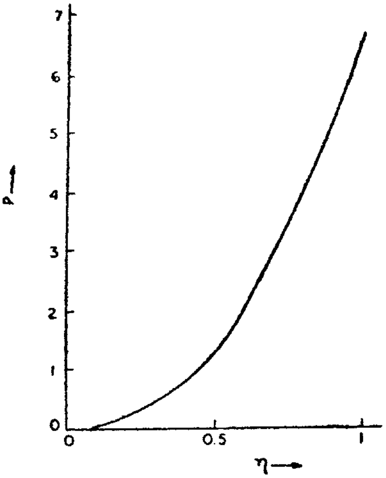

FIG. 3. Variation of density.

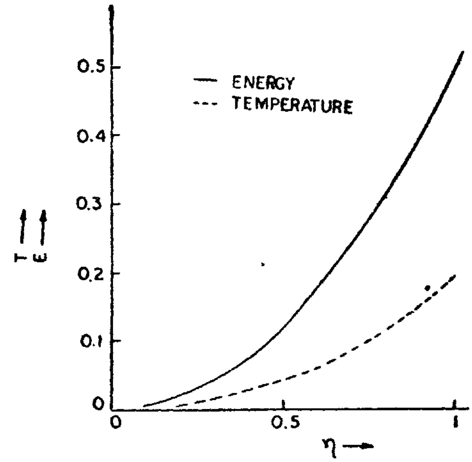

FIG. 4. Variation of temperature and energy.

the behaviour of the variables is different when the radiation pressure and energy are more effective at high temperature. Similarly the effect of magnetic field can be observed by taking different values of M_A.

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