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CREEP TRANSITIONS IN ANISOTROPIC SPHERICAL SHELLS UNDER UNIFORM INTERNAL PRESSURE

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Seth's transition theory of elastic-plastic and creep deformations has been applied to study the creep transitions in an anisotropic spherical shell subjected to uniform internal pressure. The anisotropy considered here is of spherical type. The expressions for transitional stresses are derived which lead to the creep stresses given by the classical theories of creep.

1. INTRODUCTION

The classical theories of creep start with the assumptions of constitutive equations for creep and the classical theories of plasticity need an extra relation called the yield condition in addition to the flow rules. The description of the deformations in a solid subjected to external forces is thus given by a different set of equations for elastic, plastic and creep deformations. Seth (1962, 1964) has shown that all this is not necessary if one recognizes that the plastic and creep states are reached through a transition state which is initially elastic in nature. This transition approach to the problems of plasticity and creep has been successfully applied to the large number of problems. In this paper we apply this method to the problem of creep transitions in a spherical shell under uniform internal pressure where the shell is assumed to have transverse isotropy about the radius vector. We have derived the expressions for the transitional stresses which lead to the creep stresses given by the classical theories of creep (Finnie and Heller 1959, Johnson 1977).

2. STRESS-STRAIN RELATIONS

Seth (1962, 1964) initially proposed the transition theory for the isotropic materials undergoing elastic-plastic and creep deformations. Later he (Seth 1970) extended his theory to the plastic deformations of anisotropic solids. The main step in this work consists of starting with a linear stress-strain relations and then applying the transition techniques developed for isotropic materials. Here we shall follow exactly the same path. For our problem, we shall need the stress-strain relations in spherical coordinates (r, θ, ϕ) for materials having transverse isotropy about the radius vector. The required stress-strain relations are given by (Lekhnitskii 1963, Vlaar 1969)

$$\left.\begin{aligned} T_{rr} &= c_{33}e_{rr} + c_{13}e_{\theta\theta} + c_{13}e_{\phi\phi} \\ T_{\theta\theta} &= c_{13}e_{rr} + c_{11}e_{\theta\theta} + c_{12}e_{\phi\phi} \\ T_{\phi\phi} &= c_{13}e_{rr} + c_{12}e_{\theta\theta} + c_{11}e_{\phi\phi} \\ T_{r\theta} &= c_{44}e_{r\theta},\ T_{r\phi} = c_{44}e_{r\phi},\ T_{\theta\phi} = c_{66}e_{\theta\phi} \end{aligned}\right\} \quad \text{...(2.1)}$$

where T, e denote stress and strain respectively, with usual convention on subscripts, and c_{ij} are material constants with $2c_{66} = c_{11} - c_{12}$.

It is well-known that for creep transitions in isotropic materials one has to take the Seth's generalized strain measure (Seth 1966). In terms of Almansi strain components ϵ^A_{ij}, the principal components of generalized strain measure e_{ii} (i not summed) are given by

$$e_{ii} = \left[\frac{1}{n}\left\{1 - (1 - 2\bar{\epsilon}^A_{ii})^{n/2}\right\}\right]^m \quad \text{...(2.2)}$$

where m, n are measure exponents and $\bar{\epsilon}^A_{ii}$ (i not summed) are principal components of ϵ^A_{ij}. The components of ϵ^A_{ij} are given in Appendix I. Seth has shown that this generalized strain measure, for a particular choice of m and n reduces to the other known strain measures in the literature. Following the transition methods for isotropic materials, we shall assume the stress-strain relations given by eqn. (2.1) where strains e_{ij} stand for the generalized strains given by eqn. (2.2). This method has also been applied to the creep transitions in aelotropic cylinders under uniform pressure (1978).

3. Differential Equation Governing Transitions

We consider a hollow spherical shell of constant thickness under uniform internal pressure. The material of the shell is assumed to posses the transverse isotropy about the radius vector. The symmetry in the problem allows us to take the displacements, in the obvious choice of the spherical coordinates (r, θ, ϕ), as

$$u = r(1 - p),\ v = 0,\ w = 0 \quad \text{...(3.1)}$$

where p is a function of r only. Then the Almansi strain components ϵ^A_{ij} are given by (cf. Seth 1935)

$$\left.\begin{aligned} \epsilon^A_{rr} &= \tfrac{1}{2}\{1 - (rp' + p)^2\} \\ \epsilon^A_{\theta\theta} &= \epsilon^A_{\phi\phi} = \tfrac{1}{2}(1 - p^2) \\ \epsilon^A_{r\theta} &= \epsilon^A_{\theta\phi} = \epsilon^A_{r\phi} = 0 \end{aligned}\right\} \quad \text{...(3.2)'}$$

where prime denotes differentiation with respect to r. Since shear strains are zero, we have $\bar{\epsilon}^A_{rr} = \epsilon^A_{rr}$, $\bar{\epsilon}^A_{\theta\theta} = \epsilon^A_{\theta\theta}$ and $\bar{\epsilon}^A_{\phi\phi} = \epsilon^A_{\phi\phi}$. Therefore, the generalized strains, from eqn. (2.2), are given by

$$\left.\begin{aligned} e_{rr} &= \frac{1}{n^m}\{1-(rp'+p)^n\}^m \\ e_{\theta\theta} &= e_{\phi\phi} = \frac{1}{n^m}(1-p^n)^m \\ e_{r\theta} &= e_{\theta\phi} = e_{r\phi} = 0 \end{aligned}\right\} \qquad \ldots(3.2)$$

where prime denotes differentiation with respect to r.

From the stress-strain relations given by eqn. (2.1), we get the following expressions for stresses :

$$\left.\begin{aligned} T_{rr} &= c_{33}e_{rr} + 2c_{13}e_{\theta\theta} \\ T_{\theta\theta} &= T_{\phi\phi} = 2(c_{11}-c_{66})\,e_{\theta\theta} + c_{13}e_{rr} \\ T_{r\theta} &= T_{\theta\phi} = T_{r\phi} = 0 \end{aligned}\right\} \qquad \ldots(3.3)$$

where e_{rr}, $e_{\theta\theta}$ are given by eqns. (3.2).

The equation of equilibrium

$$\frac{\partial T_{rr}}{\partial r} + \frac{2(T_{rr}-T_{\theta\theta})}{r} = 0 \qquad \ldots(3.4)$$

leads to the differential equation

$$c_{33}\{1-(rp'+p)^n\}^{m-1}(rp'+p)^{n-1}(rp''+2p') + 2c_{13}(1-p^n)^{m-1}\,.\,p^{n-1}p'$$
$$-\frac{2}{mnr}[(c_{33}-c_{13})\{1-(rp'+p)^n\}^m + 2(c_{13}-c_{11}+c_{66})$$
$$\times(1-p^n)^m] = 0. \qquad \ldots(3.5)$$

The substitutions

$$\frac{c_{33}-c_{13}}{c_{33}} = c',\ \frac{2(c_{11}-c_{66}-c_{13})}{c_{33}-c_{13}} = 1-c'', \qquad \ldots(3.6)$$

and

$$rp' = pU, \qquad \ldots(3.7)$$

reduces eqn. (3.5) to

$$\{1-p^n(U+1)^n\}^{m-1}(U+1)^{n-1}\,Up\,\frac{dU}{dp} + \{1-p^n(U+1)^n\}^{m-1}\times$$

(*equation continued on p.* 696)

$$\times (U + 1)^n U + 2(1 - c')(1 - p^n)^{m-1} U - \frac{2c'}{mnp^n}$$
$$\times [\{1 - p^n(U + 1)^n\}^m - (1 - c'')(1 - p^n)^m] = 0. \quad ...(3.8)$$

This is the differential equation which governs transitions and shows that the transition points of p are

$$U = -1, \; U = \pm\infty.$$

4. Creep Stresses

The creep transitions should occur through $(T_{rr} - T_{\theta\theta})$ at the transition point $U = -1$ by the obvious similarity with the isotropic case. Now, we have

$$T_{rr} - T_{\theta\theta} = \frac{(c_{33} - c_{13})}{n^m} [\{1 - p^n(U + 1)^n\}^m - (1 - c'')(1 - p^n)^m] \quad ...(4.1)$$

from eqns. (3.3) and (3.5).

Differentiating eqn. (4.1) with respect to p and using eqn. (3.7), we get

$$\frac{d \log (T_{rr} - T_{\theta\theta})}{dp} = 2c' \frac{d \log r}{dp} + \frac{mn(3 - 2c' - c'')(1 - p^n)^{m-1} p^{n-1}}{\{1 - p^n(U + 1)^n\}^m - (1 - c'')(1 - p^n)^m}$$

which gives

$$T_{rr} - T_{\theta\theta} = A_0 r^{-2c'} \{1 - (1 - c'')(1 - p^n)^m\}^z$$
$$(z = (3 - 2c' - c'')/(1 - c'))$$

at the transition point $U = -1$, with A_0 as a constant. The asymptotic value of p at $U = -1$ is (B_0/r) where B_0 is a constant. Hence, we have

$$T_{rr} - T_{\theta\theta} = A_0 r^{-2c'} \{1 - (1 - c'')(1 - Br^{-n})^m\}^z. \quad ...(4.2)$$

This expression is very general in nature and to get some compact results we shall consider a special case. We note that as the transition progresses the medium is undergoing changes and consequently the material constants c_{ij} will no longer have the same initial values. The values of c' and c'' will be continuously changing and will approach the value zero independently as the transition is completed. This follows from the criteria given by Seth (1976). To be specific the condition to be satisfied is $\nu_{ij} + \nu_{ji} = 1$, $i \neq j$ where ν_{ij} are generalized Poisson's ratios. Therefore, we shall consider the special case of eqn. (4.2) when $m = 0$ and $c'' \to 0$. In this case we get

$$T_{rr} - T_{\theta\theta} = Ar^{-3n+2c'(n-1)} \quad ...(4.3)$$

where A is a constant. We remark here that due to presence of c' in eqn. (4.3) the physical state of the material is still a transition state.

Equations (3.4) and (4.3) give

$$T_{rr} = \frac{A r^{-3n+2c'(n-1)}}{3n - 2c'(n-1)} + B, \qquad \ldots(4.4)$$

where B is a constant of integration.

The boundary conditions for the spherical shell under uniform internal pressure $\bar{p}$ are

$$T_{rr} = -\bar{p} \quad \text{at} \quad r = a,$$
$$T_{rr} = 0 \quad \text{at} \quad r = b, \qquad \ldots(4.5)$$

where a, b are respectively the internal and external radii of the spherical shell. These will lead to the following expressions for transitional stresses :

$$T_{rr} = -\bar{p}\,\frac{(b/r)^{\nu} - 1}{(b/a)^{\nu} - 1},$$

$$T_{\theta\theta} = T_{\phi\phi} = \bar{p}\,\frac{\frac{1}{2}y\,(b/r)^{\nu} + 1}{(b/a)^{\nu} - 1} \qquad \ldots(4.6)$$

$$[y = 3n - 2c'(n-1)].$$

Now the creep stresses can be obtained by letting $c' \to 0$ in eqn. (4.6) and thus we get

$$\left.\begin{aligned} T_{rr} &= -\bar{p}\,\frac{(b/r)^{3n} - 1}{(b/a)^{3n} - 1} \\ T_{\theta\theta} &= T_{\phi\phi} = \bar{p}\,\frac{(1/2)(3n-2)(b/r)^{3n} + 1}{(b/a)^{3n} - 1}\,. \end{aligned}\right\} \qquad \ldots(4.7)$$

These expressions are same as those for isotropic materials. This is not surprising because even in the classical theory of creep this is what happens (Finnie and Heller 1959). This can also be deduced from the recently proposed new constitutive equations for creep of transversely isotropic materials by Johnson (1977).

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Appendix I

The components of ϵ_{ij}^{A} in spherical coordinates (r, θ, ϕ) with displacements u, v, w along the coordinate curves are given by

$$2\epsilon_{rr}^{A} = 1 - \left(1 - \frac{\partial u}{\partial r}\right)^2 - \left(\frac{\partial v}{\partial r}\right)^2 - \left(\frac{\partial w}{\partial r}\right)^2,$$

$$2\epsilon_{\theta\theta}^{A} = 1 - \left\{1 - \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)\right\}^2 - \left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 - \left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2,$$

$$2\epsilon_{\phi\phi}^{A} = 1 - \left\{1 - \left(\frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{u}{r} + \frac{\cot\theta}{r}v\right\}^2 - \left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{w}{r}\right)^2 - \left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} - \frac{\cot\theta}{r}w\right)^2,$$

$$2\epsilon_{r\theta}^{A} = \left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) - \frac{\partial u}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) - \frac{\partial v}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right),$$

$$2\epsilon_{r\phi}^{A} = \left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r}\right) - \frac{\partial u}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{w}{r}\right) - \frac{\partial v}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} - \frac{\cot\theta}{r}w\right) - \frac{\partial w}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{u}{r} + \frac{\cot\theta}{r}w\right),$$

$$2\epsilon_{\theta\phi}^{A} = \left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} + \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\cot\theta}{r}w\right) - \frac{1}{r}\frac{\partial w}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{u}{r} + \frac{\cot\theta}{r}v\right) - \left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{w}{r}\right) - \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)\left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} - \frac{\cot\theta}{r}w\right).$$