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A GALERKIN ALGORITHM FOR SOLVING CAUCHY-TYPE SINGULAR INTEGRAL EQUATIONS

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In a recent paper Delves *et al.* (1979) described a modified Galerkin algorithm using a Chebyshev expansion for $f(x)$:

$$f(x) \approx \sum_{i=0}^{N} a_i T_i(x)$$

to solve weakly singular integral equations of the second kind, with operation count $O(N^2 \ln N)$ for setting up equations and $O(N^3)$ for solving the linear equations defining the coefficients a_i, $i = 0, 1, \ldots, N$. Also the algorithm provides an error estimate which is easy to compute since it relates the estimate to quantities which are available during the course of the calculation. The author describes here an extension of this algorithm to include Cauchy-type singular integral equations.

1. INTRODUCTION

The principal goal here is to show that the integral equation

$$f(x) = g(x) + \int_{-1}^{1} \frac{K(x, y) f(y) \, dy}{(y - x)} \qquad \ldots(1.1)$$

can be solved numerically using the Galerkin technique based on Chebyshev expansion. The integral in (1.1) can be understood as a principal value integral. Delves *et al.* (1979) used a modified Galerkin technique and efficiently solved

$$f(x) = g(x) + \int_{-1}^{1} e(x, y) \, Q(x, y) \, f(y) \, dy \qquad \ldots(1.2)$$

singular integral equation, where e is a smooth function and Q contains known logarithmic or weak singularities. In this method a Chebyshev approximation for the solution $f(x)$:

$$f \approx f_N(x) = \sum_{i=0}^{N} \bar{a}_i T_i(x) \qquad \ldots(1.3)$$

is used. Equation (1.2) is solved with operation count $O(N^2 \ln N)$, while all the methods described so far including the standard Galerkin method proposed operation count of $O(N^3)$. The method also yields an efficient and computable error estimate.

In this paper we develop two different methods I and II which convert eqn. (1.1) into another with kernels of type $w(x, y) \ln (y - x)$ and hence it can be treated using the technique adopted by Delves *et al.* (1979). Description for the use of methods I and II in solving eqn. (1.1) is given in section 2, while we refer to Delves *et al.* (1979) in providing the fast algorithm to evaluate numerically the elements of the linear system and the quadrature error accompanied with this approximation. The iterative procedure (Delves 1977b) is used in solving the linear system of equations defining the coefficients a_i, $i = 0, 1, ..., N$. The error in the numerical solution obtained by the algorithm considered in this paper has been discussed before by Delves *et al.* (1979) and Delves (1977a). This discussion remains valid subject to appropriate modification to suit the two different methods considered in this paper, and so a brevi discussion for the error analysis is given in section 3. To demonstıate the effectiveness of both methods in treating Cauchy's singularity, a numerical example is solved using methods I and II. Method II, not only treats Cauchy's singularity of variable type as in eqn. (1.1), but also of fixed type as:

$$f(x) = g(x) + \int_{-1}^{1} \frac{K(x, y) f(y) \, dy}{(y - q)}, \quad -1 < q < 1. \qquad ...(1.4)$$

2. Description of the Two Methods

In eqn. (1.1), the singular function $1/(y - x)$ has no Chebyshev expansion, so we cannot use algorithm (Delves *et al.* 1979) to treat the singularity. In this section, we consider two different methods to convert eqn. (1.1) into another with kernels bearing logarithmic singularities, then we solve it using the modified Galerkin technique (Delves *et al.* 1979).

Method I

This method depends on integrating eqn. (1.1) and hence the integrated form of this equation is

$$\int_{-1}^{x} f(t) \, dt = \int_{-1}^{x} g(t) \, dt + \int_{-1}^{1} dy \, f(y) \, M(x, y) \qquad ...(2.1)$$

where

$$M(x, y) = \ln (y + 1) K(-1, y) - \ln (y - x) K(x, y) + \int_{-1}^{x} \ln (y - t) K_t(t, y) \, dt \qquad ...(2.2)$$

Using Galerkin technique, approximating $f(x)$ by (1.3) and $\int_{-1}^{x} f(t)\,dt$ by

$$f^*(x) = \int_{-1}^{x} f(t)\,dt \approx \sum_{j=0}^{N+1} \bar{a}_j^*\, T_j(x) \qquad \text{...(2.3)}$$

where Chebyshev coefficients $\bar{a}_j$ and $\bar{a}_j^*$ defined by

$$\bar{a}_j = \lambda \int_{-1}^{1} f(x)\, T_j(x)\,(1 - x^2)^{-1/2}\,dx$$

$$\bar{a}_j^* = \lambda \int_{-1}^{1} f^*(x)\, T_j(x)\,(1 - x^2)^{-1/2}\,dx$$

where $\lambda = 1/\pi$ for $j = 0$ and $\lambda = 2/\pi$ for $j > 0$; and under the assumptions :

$$g^*(x) = \int_{-1}^{x} g(t)\,dt = {\sum_{j=0}^{\infty}}' \bar{g}_j^*\, T_j(x) \qquad \text{...(2.4)}$$

$$b(y) = \ln(y + 1)\, K(-1, y)\,(1 - y^2)^{1/2} = {\sum_{j=0}^{\infty}}' b_j T_j(y) \qquad \text{...(2.5)}$$

$$A^*(x, y) = (1 - y^2)^{1/2} \int_{-1}^{x} \ln(y - t)\, K_t(t, y)\,dt = {\sum_{i,j=0}^{\infty}}' \bar{A}_{ij}^*\, T_i(x)\, T_j(y) \qquad \text{...(2.6)}$$

$$B(x, y) = K(x, y) \ln(y - x)\,(1 - y^2)^{1/2} = {\sum_{i,j=0}^{\infty}}' \bar{B}_{ij} T_i(x)\, T_j(y) \qquad (2.7)$$

Σ' denotes a sum with first term halved. Equation (2.1) reduces to

$$\sum_{j=0}^{N+1} \bar{a}_j^*\,(1 + \delta_{i0})\,\delta_{ij} - \sum_{j=0}^{N} \bar{a}_j \bar{M}_{ij} = \bar{g}_i^* \quad i = 0, 1, \ldots, N \qquad \text{...(2.8)}$$

where $$\bar{g}_i^* = \frac{2}{\pi} \int_{-1}^{1} dx\, T_i(x)\, g^*(x)\,(1 - x^2)^{-1/2}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

also,

$$\bar{M}_{ij} = \pi \bar{b}_j \delta_{i0} - \frac{\pi}{2} \bar{B}_{ij} + \frac{\pi}{2} \bar{A}^*_{ij} \qquad \text{...(2.9)}$$

where the Chebyshev coefficients

$$\bar{b}_j = \frac{2}{\pi} \int_{-1}^{1} b(x)\, T_j(x)\, (1 - x^2)^{-1/2}\, dx \qquad \text{...(2.10)}$$

$$\bar{B}_{ij} = \frac{4}{\pi^2} \int_{-1}^{1} dx\, T_i(x)\, (1 - x^2)^{-1/2} \int_{-1}^{1} B(x, y)\, T_j(y)\, (1 - y^2)^{-1/2}\, dy \qquad \text{...(2.11)}$$

$$\bar{A}^*_{ij} = \frac{4}{\pi^2} \int_{-1}^{1} dx\, T_i(x)\, (1 - x^2)^{-1/2} \int_{-1}^{1} A^*(x, y)\, T_j(y)\, (1 - y^2)^{-1/2}\, dy.$$

Using the relation given in Fox and Parker (1968) between the coefficients $\bar{a}_j$ and $\bar{a}^*_j$, and substituting the expressions for $\bar{a}^*_j$ into the left-hand side of eqn. (2.8), and after certain rearrangements we achieve the following equality:

$$\sum_{j=0}^{N+1} \bar{a}^*_j\, (1 + \delta_{i0})\, \delta_{ij} = \sum_{j=0}^{N} \bar{a}_j \bar{D}_{ij} \quad i = 0, 1, \ldots, N \qquad \text{...(2.12)}$$

where

$$\bar{D}_{ij} = \begin{cases} (1 + \delta_{i0})\, (\delta_{i0} + \delta_{i1}), & j = 0 \\ (1 + \delta_{i0})\, (-\delta_{i0} + \delta_{i2})/4, & j = 1 \\ (1 + \delta_{i0})\, [(\delta_{i0}(-1)^j + \delta_{i,j+1})/2(j + 1) - (\delta_{i0}(-1)^j & \\ \qquad + \delta_{i,j-1})/2(j - 1)], & 2 \leqslant j \leqslant N. \end{cases} \qquad \text{...(2.13)}$$

Hence, the coefficients $\bar{a}_i$, $i = 0, 1, \ldots, N$ can be computed as the solution of matrix equation

$$[\bar{D} - \bar{M}]\, \bar{\mathbf{a}} = \bar{\mathbf{g}}^*. \qquad \text{...(2.14)}$$

To construct the vector $\bar{g}^*$, first, construct the vector $\bar{g}$ with elements $\bar{g}_i$, $i = 0, 1, \ldots, N$ defined by

$$\bar{g}_i = \frac{2}{\pi} \int_{-1}^{1} g(x)\, T_i(x)\, (1 - x^2)^{-1/2}\, dx. \qquad \text{...(2.15)}$$

For $i = 0, 1, \ldots, N$, approximate values g_i are generated for $\bar{g}_i$ using the fast algorithm Delves *et al.* (1979) in operation count $O(N \ln N)$. Second, since $\bar{g}_i$ of (2.15) are Chebyshev coefficients for Chebyshev expansion:

$$g(x) = {\sum_{j=0}^{\infty}}' \bar{g}_j T_j(x) \qquad \text{...(2.16)}$$

the approximate values g_i^* for $\bar{g}_i^*$, $i = 0, 1, \ldots, N$, are achieved using relation between g_i and g_i^*. The quadrature error in the vector g represented by $\delta \underline{g} = \bar{\underline{g}} - \underline{g}$. We refer to Delves *et al.* (1979) to get an estimate for $\| \delta \underline{g} \|_\infty$ and hence $\| \delta \underline{g}^* \|_\infty$ since

$$\| \delta \underline{g} \|_\infty = \| \delta \underline{g}^* \|_\infty.$$

To construct the approximate matrix M for matrix $\bar{M}$ of eqn. (2.14), we first need to evaluate numerical values b_j, B_{ij}, A_{ij} to the elements $\bar{b}_j$, $\bar{B}_{ij}$, $\bar{A}_{ij}$ where $\bar{b}_j$ and $\bar{B}_{ij}$ defined by eqns. (2.10) and (2.11) while $\bar{A}_{ij}$ are Chebyshev coefficients for Chebyshev expansions

$$A(x, y) = \ln (y - x)\, K_x(x, y)\, (1 - y^2)^{1/2} = {\sum_{i,j=0}^{\infty}}' \bar{A}_{ij} T_i(x)\, T_j(y) \qquad \text{...(2.17)}$$

and defined by

$$\bar{A}_{ij} = \frac{4}{\pi} \int_{-1}^{1} dx\, T_i(x)\, (1 - x^2)^{-1/2} \int_{-1}^{1} A(x, y)\, T_j(y)\, (1 - y^2)^{-1/2}\, dy. \qquad \text{...(2.18)}$$

To evaluate the integrals (2.10, 2.11, 2.18) numerically we refer to the fast algorithm (Delves *et al.* 1979). In this algorithm the coefficients b_j, $j = 0, 1, \ldots, N$ are written in terms of Chebyshev coefficients for both functions $K(-1, y)$ and $\ln (y + 1)(1 - y^2)^{1/2}$, while B_{ij} and A_{ij}, $i, j = 0, 1, \ldots, N$ are written in terms of Chebyshev coefficients for functions $K(x, y)$, $\ln (y - x)\, (1 - y^2)^{1/2}$ and Chebyshev coefficients for functions $K_x(x, y)$, $\ln (y - x)\, (1 - y^2)^{1/2}$ respectively. While Chebyshev coefficients for functions $\ln (y + 1)\, (1 - y^2)^{1/2}$ and $\ln (y - x)\, (1 - y^2)^{1/2}$ are evaluated exactly, Chebyshev coefficients for $K(-1, y)$, $K(x, y)$ and $K_x(x, y)$ are given numerically using fast Fourier transform technique. The fast algorithm leads to small quadrature error in spite of the singularities in the integrands. Also it takes operation count $O(N^2 \ln N)$ to evaluate B_{ij}, A_{ij} and $O(N \ln N)$ to evaluate b_j. $i, j = 0, 1, \ldots, N$.

Using relation between A^*_{ij} and A_{ij}, numerical values for A^*_{ij} can be directly achieved and hence numerical values M_{ij} for $\bar{M}_{ij}$, $i, j = 0, 1, \ldots, N$ of eqn. (2.9) are obtained.

The quadrature error in the matrix M represented by $\delta M = \bar{M} - M$ is estimated by

$$\| \delta M \|_\infty \sim \left(\pi \, \| \delta \underline{\mathbf{b}} \|_\infty + \frac{\pi}{2} \, \| \delta B \|_\infty + \frac{\pi}{2} \, \| \delta A \|_\infty \right) \qquad \ldots(2.20)$$

where $\| \delta \underline{\mathbf{b}} \|_\infty$, $\| \delta A \|_\infty$, $\| \delta B \|_\infty$ are given by (1).

Method II

Integrating eqn. (1.1) by parts, we achieve:

$$f(x) - \ln(1 - x)\, K(x, 1)\, f(1) + \{\pi \hat{i} + \ln(1 + x)\}\, K(x, -1)\, f(-1)$$

$$= g(x) - \int_{-1}^{1} \ln(y - x)\, K(x, y)\, f'(y)\, dy$$

$$- \int_{-1}^{1} \ln(y - x)\, K_y(x, y)\, f(y)\, dy \qquad \ldots(2.21)$$

where $\hat{i} = \sqrt{-1}$.

Applying the Galerkin technique and using Chebyshev expansion (1.3) for $f(x)$, and for $f'(x)$:

$$f'(x) \approx \sum_{j=0}^{N-1} \bar{a}^\dagger_j \, T_j(x) \qquad \ldots(2.22)$$

where $\qquad \bar{a}^\dagger_j = \lambda \int_{-1}^{1} dx\, T_j(x)\, f'(x)\, (1 - x^2)^{-1/2}$

and under the assumptions:

$$K(x, -1) = {\sum_{j=0}^{\infty}}' \bar{k}_j T_j(x) \qquad \ldots(2.23)$$

$$p(x) = K(x, 1) \ln (1 - x) = \sum_{j=0}^{\infty}{}' \bar{p}_j T_j(x) \qquad ...(2.24)$$

$$q(x) = K(x, -1) \ln (1 + x) = \sum_{j=0}^{\infty}{}' \bar{q}_j T_j(x) \qquad ...(2.25)$$

$$C(x, y) = K_y(x, y) \ln (y - x) (1 - y^2)^{1/2} = \sum_{i,j=0}^{\infty}{}' \bar{C}_{ij} T_i(x)\, T_j(y) \qquad ...(2.26)$$

eqn. (2.21) reduces to

$$\sum_{j=0}^{N} \bar{a}_j \bar{Q}_{ij} + \frac{\pi}{2} \sum_{j=0}^{N-1} \bar{a}_j^{\dagger}\, \bar{B}_{ij} = \bar{g}_i, \quad i = 0, 1, ..., N \qquad ...(2.27)$$

where

$$\bar{Q}_{ij} = (1 + \delta_{i0})\, \delta_{ij} - \bar{p}_i + \pi \hat{i} (-1)^j\, \bar{k}_i + \frac{\pi}{2}\, \bar{C}_{ij} + (-1)^j\, \bar{q}_i \qquad ...(2.28)$$

$\bar{B}_{ij}$, $\bar{g}_i$ are defined by eqns. (2.11) and (2.15) respectively.

The relation between $\bar{a}_j^{\dagger}$ and $\bar{a}_j$ leads to

$$\sum_{j=0}^{N-1} \bar{a}_j^{\dagger}\, \bar{B}_{ij} = \sum_{j=1}^{N} 2j\bar{a}_j \sum_{r=0}^{[(j-1)/2]} \bar{B}_{i,A} \qquad ...(2.29)$$

where $[(j - 1)/2]$ denotes the integral part of $(j - 1)/2$; Σ'' means that the term which has the subscript $A = 2r + 2\left(\frac{j-1}{2} - \left[\frac{j-1}{2}\right]\right) = 0$ is halved. Hence the set of eqns. (2.27) reduces to

$$[\bar{\Delta} - \bar{W}]\, \bar{\mathbf{a}} = \bar{\mathbf{g}} \qquad ...(2.30)$$

where $\bar{\Delta}$ and $\bar{W}$ are square matrices of order $(N + 1)$ with elements

$$\bar{\Delta}_{ij} = (1 + \delta_{i0})\, \delta_{ij}, \quad i, j = 0, 1, ..., N \qquad ...(2.31)$$

and

$$\bar{W}_{ij} = \bar{Q}_{ij} - \pi j \sum_{r=0}^{[(j-1)/2]}{}'' \bar{B}_{i,A}, \quad i, j = 0, 1, ..., N. \qquad ...(2.32)$$

Here subscript $A = 2r + 2\left(\frac{j-1}{2} - \left[\frac{j-1}{2}\right]\right)$.

To solve the linear system (2.30) in the unknown vector $\bar{a}$, we need to evaluate numerically the elements of the vector $\bar{g}$ and matrix $\bar{W}$. The approximate values B_{ij} and g_i are discussed in Method I. The approximations p_i, q_i and C_{ij} of the elements $\bar{p}_i$, $\bar{q}_i$ and $\bar{C}_{ij}$ defined by

$$\bar{p}_i = \frac{2}{\pi} \int_{-1}^{1} T_i(x)\, p(x)\, (1 - x^2)^{-1/2}\, dx \qquad \text{...(2.33)}$$

$$\bar{q}_i = \frac{2}{\pi} \int_{-1}^{1} T_i(x)\, q(x)\, (1 - x^2)^{-1/2}\, dx \qquad \text{...(2.34)}$$

$$\bar{C}_{ij} = \frac{4}{\pi^2} \int_{-1}^{1} dx\, T_i(x)\, (1 - x^2)^{-1/2} \int_{-1}^{1} C(x, y)\, T_j(y)(1 - y^2)^{-1/2}\, dy \qquad \text{...(2.35)}$$

can be achieved using the fast algorithm (Delves *et al.* 1979). Hence the quadrature error in the matrix W represented by $\delta W = \bar{W} - W$, is estimated by

$$\| \delta W \|_\infty \simeq (\| \delta \mathbf{p} \|_\infty + \pi \| \delta \mathbf{k} \|_\infty + \| \delta \mathbf{q} \|_\infty + \frac{\pi}{2} \| \delta C \|_\infty + \pi N \| \delta B \|_\infty) \qquad \text{...(2.36)}$$

where $\| \delta p \|_\infty$, $\| \delta k \|_\infty$, $\| \delta q \|_\infty$, $\| \delta C \|_\infty$, $\| \delta B \|_\infty$ are given by (1).

Solution of Equations

The linear system of eqns. (2.14) and (2.30) which we put in the general form

$$\bar{L}\bar{a} = \bar{\mathbf{r}} \qquad \text{...(2.37)}$$

where $\bar{L} = \bar{D} - \bar{M}$, and $\bar{r} = \bar{\mathbf{g}}^*$ in case of eqn. (2.14), while $\bar{L} = \bar{\Delta} - \bar{W}$ and $\bar{\mathbf{r}} = \bar{\mathbf{g}}$ in case of eqn. (2.30); is solved iteratively using the procedure discussed in Delves (1977b) with a further simplification that M and W are known explicitly.

3. Error Analysis

The error in the numerical solution $e_N(x) = f(x) - f_N(x)$ has three components:

(1) The truncation error stemming from truncating the series (1.3) at the Nth term and according to results given in Delves (1977a) a suitable estimate for it is $N\, a_N$, where a_N is the computed estimate for $\bar{a}_N$.

(2) The quadrature error due to the numerical estimation of the matrix L and vector r, hence eqn. (2.37) can be written in the form

$$(L + \delta L)(a + \delta a) = r + \delta r \qquad ...(3.1)$$

and hence

$$\| \delta \mathbf{a} \| \leqslant \| L^{-1} \| (\| \delta L \| \| \mathbf{a} \| + \| \delta \mathbf{r} \|)/(1 - \| L^{-1} \| \| \delta L \|) \qquad ...(3.2)$$

where $\mathbf{a}$ is the computed vector for $\bar{\mathbf{a}}$. From (2), a rough estimate for $\| L^{-1} \|$ can be taken to be

$$\| L^{-1} \| \sim \| \mathbf{r} \| / \| \mathbf{a} \| \qquad ...(3.3)$$

also $\| \delta r \| = \| \delta g \|$, and $\| \delta L \| = \| \delta M \|$ for Method I, and $\| \delta W \|$ for Method II. Equations (2.20) and (2.36) give a numerical estimation for $\| \delta M \|$ and $\| \delta W \|$ respectively.

(3) The error due to the iterative solution of equations and it is arbitrarily small and so we neglect it. Hence

$$| e_N | \sim N a_N + \| \delta a \|. \qquad ...(3.4)$$

4. Numerical Example

In this section, we give the numerical results obtained by solving eqn. (1.1) using Methods I and II, where

$$g(x) = [1 + 2\pi i x - 2x \ln \{(1 - x)/(1 + x)\} - (2 - x) \ln 3]/(x + 2)$$

$$K(x, y) = (x + y).$$

The exact solution $f(x) = 1/(x + 2)$.

The computed error σ_N is given to be

$$\sigma_N = \left\{ \sum_{j=0}^{N} | e_N^2 (x_j) | /N \right\}^{1/2} \approx \left\{ \int_{-1}^{1} | e_N^2 (x) | \, dx \right\}^{1/2}$$

where $x_j = \cos (j\pi/N)$, $j = 0, 1, ..., N$. We use eqn. (3.4) as an estimate for $| e_N |$ in evaluating σ_N (est. error).

The numerical results obtained (Table I) reflect the convergence of both the Methods I and II. They emphasise that we can successfully treat Cauchy's singularity in the kernel. The error estimates obtained reflect the actual error extremely well. The results achieved from Method I are little better than those for Method II, which is to be expected since this integration technique often has a good effect on the accuracy. While $\| \delta M \|_\infty$ include the term $\| \delta B \|_\infty$, $\| \delta W \|_\infty$ include the term

$N \| \delta B \|_\infty$; also Fox (1962) used the method of "prior integration" to get better accuracy with Lanczos method.

TABLE I

Computed Results for the Numerical Example

N	Method I		Method II	
	σ_N	est. error	σ_N	est. error
2	9.1×10^{-1}	3.2×10^{0}	2.3×10^{-1}	8.2×10^{0}
3	8.4×10^{-2}	3.7×10^{-1}	4.8×10^{-2}	1.6×10^{-1}
4	5.0×10^{-3}	4.2×10^{-2}	4.6×10^{-3}	3.4×10^{-2}
5	8.7×10^{-5}	4.3×10^{-4}	3.7×10^{-4}	1.6×10^{-3}
6	7.5×10^{-6}	2.3×10^{-5}	2.5×10^{-5}	3.7×10^{-4}
7	6.0×10^{-7}	4.3×10^{-6}	1.4×10^{-6}	1.6×10^{-5}
8	4.6×10^{-8}	4.1×10^{-7}	7.6×10^{-8}	8.2×10^{-7}
9	5.1×10^{-9}	4.8×10^{-8}	6.1×10^{-9}	6.2×10^{-8}
10	4.4×10^{-10}	1.2×10^{-9}	7.1×10^{-10}	5.6×10^{-9}

5. COMMENTS ON THE ALGORITHM

(1) If $K(x, y) = 1$, then $\bar{A}_{ij} = \bar{C}_{ij} = 0$, and all the Chebyshev coefficients appearing as elements of matrix M or W, can be evaluated exactly. Hence

$$\delta M = \delta W = 0.$$

(2) Although Methods I and II, are described to solve integral equations with Cauchy's singularity of variable type ($y = x$) as given in (1.1), Method II can also be used to solve equations of the type given by (1.4) where the singularity is fixed ($y = q$).

(3) if the integral equation has the form

$$f(x) = g(x) + \int_{-1}^{1} K(x, y)\, f(y)\, dy \qquad \ldots(5.1)$$

where $K(x, y)$ is regular, while the solution of the equation has the form

$$f(y) = f_r(y)/(y - q), \quad -1 < q < 1 \qquad \ldots(5.2)$$

where $f_r(y)$ is regular, then the Chebyshev expansion of the solution f is slowly convergent. To overcome this difficulty, we write eqn. (5.1) as an equation for f_r:

$$f_r(x) = g_*(x) + \int_{-1}^{1} dy\, K_*(x, y)\, f_r(y)/(y - q), \quad -1 < q < 1 \qquad \ldots(5.3)$$

where $K_*(x, y) = (x - q)\, K(x, y)$ and $g_*(x) = (x - q)\, g(x)$.

Hence Method II can be used to solve eqn. (5.3).

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