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RIEMANNIAN 5-REGULAR MANIFOLDS AS COSET MANIFOLDS

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In this paper the relation between a Riemannian 5-regular symmetric manifold M and the homogeneous space G/H is found, where G is the identity component of the Lie group of all almost complex isometries on M and H is the compact subgroup of G which leaves a fixed point of M fixed.

1. INTRODUCTION

In Helgason (1964) the following result is proved.

Let G be a Lie group which acts transitively on a manifold M. Let H be the isotropy subgroup of a fixed point $P \in M$. Then H is a closed and G/H is diffeomorphic to M under the map

$$f : G/H \to M$$

given by $f(gH) = g.p, g \in G$ and $p \in M$.

Definition 1 — Let (M, g) be a Riemannian manifold, if for each point $P \in M$, there exists a neighbourhood U_p of P in M and a local symmetry S_p of P such that S_p is an isometry of U_p, then M is called locally symmetric manifold.

In Kobayashi and Namizu (1963), it was proved that the set $I(M)$ of all isometries acting on a locally Riemannian symmetric manifold is a transitive Lie transformation group. Hence it is diffeomorphic to G/H where G is the identity component (connected) of $I(M)$ and H is the compact subgroup of G which leaves a fixed point of M fixed. In Hausner and Schwartz (1968) it is proved that a closed subgroup of a Lie transformation group is a Lie transformation group.

Definition 2 — A Riemannian locally 3-symmetric manifold is a manifold with a family $C(M)$ of 3-order isometries $P \to S_p$ each of which is an almost complex isometry of the family.

Gray (1971) proved that the family of 3-order isometries $C(M)$ acting on a Riemannian locally 3-symmetric manifold is a Lie transformation group, hence the manifold is diffeomorphic to G/H where G is the identity component of $C(M)$ and H is the compact subgroup of G which leaves a fixed point of M fixed.

Unless otherwise stated, all manifolds, maps, vector fields and tensor fields are of class C^∞.

2. The Riemannian Locally 5-Symmetric Manifold Case

Definition 3 — A Riemannian locally 5-symmetric manifold is a manifold with a family $C(M)$ of 5-order isometries $P \to S_p$ each of which is an almost complex isometry of the family. S_p is called a symmetry at p.

Al-Aqeel (1977) proved that there is an almost complex structure J on M associated with the family $C(M)$. Ledger and Obata (1968) proved that the set $I(M)$ of all isometries acting on a locally 5-symmetric manifold is a transitive Lie transformation group of M. A Riemannian locally 5-symmetric manifold is said to be a Riemannian 5-symmetric manifold if each S_p is a global isometry.

Definition 4 — A Riemannian locally 5-symmetric manifold is said to be regular if $df \circ J = J \circ df$ where $f \in I(M)$.

Theorem 1 — The set $C(M)$ of the group of all almost complex isometries on a Riemannian 5-regular symmetric manifold is a transitive Lie transformation group of M.

Proof : For all $P \in M$, $S_p \in C(M)$ and since we only need symmetries to prove the transitivity of $C(M)$ (see Ledger and Obata 1968), we conclude that $C(M)$ is transitive on M.

Let $\{f_n\}$ be a sequence of almost complex isometries which converges to f in $I(M)$, where $I(M)$ is the set of all isometries on M. The associated almost complex structure J is continuous. We have $df_n \circ J = J \circ df_n$ for all n. From the continuity of J we see that $df \circ J = J \circ df$, and $f \in C(M)$. Hence $C(M)$ is closed in $I(M)$ and $I(M)$ is a Lie transformation group in M, this implies that $C(M)$ is a Lie transformation group of M.

If G is the largest connected component of $C(M)$ i.e. the identity component of $C(M)$ and H is the compact subgroup of G which leaves a fixed point of M fixed, then M is diffeomorphic to G/H.

Theorem 2 — Let M be a Riemannian 5-regular symmetric manifold. Then

(i) The 5-symmetry S_p, $P \in M$ induced a 5-automorphism ϕ of G, defined by

$$\phi(g) = S_p \circ g \circ S_p^{-1} \text{ for all } g \in G.$$

(ii) If H_ϕ is the subgroup of G of fixed points of ϕ, then

$$(H_\phi)_0 \subseteq H \subseteq H_\phi$$

where $(H_\phi)_0$ is the identity component of H_ϕ. Also H contains no normal subgroups of G other than $\{e\}$.

Proof : $\phi : C(M) \to C(M)$ is an automorphism of $C(M)$, and since it maps connected components to connected components, it is also automorphism of G. Now

$$\phi^2(g) = \phi[\phi(g)] = \phi(S_p \circ g \circ S_p^{-1}) = S_p^2 \circ g \circ S_p^{-2}$$

$$\phi^5(g) = \phi[\phi^4(g)] = S_p^5 \circ g \circ S_p^{-5} = g$$

which proves that ϕ is a 5-automorphism of G.

Let $h \in H$, then at $P \in M$ we have

$$[d(\phi(h))]_p = [dS_p \circ dh \circ dS_p^{-1}]_p$$

$$\circ \; dh \circ [((\cos \phi_1) I + (\sin \phi_1) J_1) \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2)]_p \circ dh \circ [((\cos \phi_1) I + (\sin \phi_1) J_1) \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2]_p^{-1}$$

$$= [((\cos \phi_1) I + (\sin \phi_1) J_1) \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2]_p \circ dh \circ [(\cos \phi_1) I + (\sin \phi_1) J_1)^{-1} \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2)^{-1}]_p$$

$$= [((\cos \phi_1) I + (\sin \phi_1) J_1) \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2]_p \circ dh \circ [((\cos \phi_1) I_1 + (\sin \phi_1)^4 \oplus ((\cos \phi_2) I + (\sin \phi_2) J_2)^4]_p$$

$$= [dh]_p.$$

Since $h \in C(M)$ i.e. $dh \circ J = J \circ dh$

Also $\phi(h)\,(p) = (S_p \circ h \circ S_p^{-1})\,(p) = p$

$\therefore \quad \phi(h) = h$

$\because \quad H \subset H.$

Let $t \to g_t$, $t \in R$, be a 1-parameter subgroup of H_ϕ. Then $\phi(g_t) = g_t$. Also $(S_p \circ g_t)\,(p) = (g_t \circ S_p)\,(p) = g_t(p)$. Hence the orbit $\{g_t(p) \mid t \in R\}$ is fixed by S_p for all $t \in R$. But p is an isolated fixed point of S_p. This means that $\{g_t(p) \mid t \in R\}$ must reduce to p. Hence $g_t \in H$, but g_t is a 1-parameter subgroup of H_ϕ and $(H_\phi)_0$ is the identity component of H. This implies that $(H_\phi)_0 \subset H$.

Let T be a normal subgroup of G in H, let $g \in G$. Then for each $k \in T$, there exists $k' \in T$ such that $k'g = gk$. Hence $k'g(p) = gk(p) = g(p)$ for all $g \in G$; i.e. if $x \in M$, and since G is transitive on M, there exists $g'\, G$ such that $g'(p) = x$, and we have $k' \cdot g'(p) = k'x = g'(p) = x$. But G acts effectively on M, so $k' = e$, and therefore $T = \{e\}$.

Theorem 3 — Let M be a Riemannian 5-regular symmetric manifold and let $\mathbf{g}$ and $\mathbf{h}$ be the Lie algebras of G and H respectively. If $\phi : G \rightarrow G$ given by $\phi(g) = S_p \circ g \circ S_p^{-1}$. Then

$$\mathbf{h} = \{X \in g \mid (d\phi)_e X = X\}$$

and if we have

$$\mathbf{m} = \{X \in \mathbf{g}\} \mid X + (d\phi)_e X + \ldots + (d\phi)_e^4 X = 0\}, \text{ then}$$

$$\mathbf{g} = \mathbf{m} \oplus \mathbf{h} \text{ (direct sum).}$$

Let $\pi : G \rightarrow M$ be the natural map given by $g \rightarrow g(p)$, then $(d\phi)_e$ map $\mathbf{h}$ into $\{\mathbf{0}\}$ and $\mathbf{m}$ isomorphically onto M_p.

Proof : $\phi : G \rightarrow G$ is an automorphism of order 5, hence $(d\phi)_e : \mathbf{g} \rightarrow \mathbf{g}$ is an automorphism of order 5 i.e.

$$(d\phi)_e^5 - I = 0.$$

Consider the polynomial

$$f(t) = t^5 - 1 = (t - 1)(t^4 + t^3 + t^2 + t + 1) = g_1(t)\, g_2(t)$$

where $g_1(t)$ and $g_2(t)$ are relatively prime.

Also

$$f[(d\phi)_e] = 0$$

$$\therefore \quad \mathbf{g} = h \oplus \mathbf{m}$$

where

$$\mathbf{h} = \text{kernel } g_1[(d\phi)_e] = \{X \in \mathbf{g} \mid (d\phi)_e X = X\}$$

and

$$\mathbf{m} = \text{kernel } g_2[(d\phi)_e] = \{X \in \mathbf{g} \mid X + (d\phi)_e X + \ldots + (d\phi)_e^4 X = 0\}$$

The map $\pi : G \rightarrow M$ maps H onto p and therefore $\mathbf{h}$ kernel $(d\pi)_e$.

Let $X \in$ kernel $(d\pi)_e$, then if $g \in C^\infty(M)$ we have

$$0 = ((d\pi)_e X)(g) = X(g \circ \pi) = \left\{\frac{d}{dt} g(\exp(tX)\cdot p)\right\}_{t=0}.$$

Let $s \in R$ and consider the function

$$g^*(q) = g(\exp(sX)\cdot q);\; q \in M. \quad \text{Then}$$

$$0 = \left\{\frac{d}{dt} g^*(\exp(tX)\cdot p)\right\}_{t=0} = \left\{\frac{d}{dt} g(\exp(tX)\cdot p)\right\}_{t=0}$$

which shows that $g(\exp(sX)\cdot p)$ is constant in s.

g is arbitrary and we have

$(\exp(sX))(p) = p$ for all $s \in R$, and so $X \in \mathbf{h}$. Hence $(d\pi)_e$ vanishes on $\mathbf{h}$. So

$$\text{rank } \pi = (\text{dimension } \mathbf{g} - \text{dimension } \mathbf{h})$$

$$= (\text{dimension } G - \text{dimension } H)$$

$$\therefore \quad \text{rank } \pi = \text{dimension } G/H = \text{dimension } M$$

Hence $(d\pi)_e$ maps $\mathbf{m}$ isomorphically onto M_p.

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